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ON UPPER FUNCTIONS FOR ANOMALOUS DIFFUSIONS GOVERNED BY TIME-VARYING ORNSTEIN–UHLENBECK PROCESS*

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Abstract. We obtain upper functions that serve as almost sure asymptotic upper bounds for a displacement process given by an integrated time-varying Ornstein–Uhlenbeck process. The form of upper functions depends on the characteristics (the stability rate and the diffusion coefficient) of a stochastic linear differential equation. We introduce the notion of anomalous diffusion related to behavior of upper functions and compare the results of diffusion classification (normal diffusion, subdiffusion, and superdiffusion) with those obtained on the basis of mean square displacements.

Key words. time-varying Ornstein–Uhlenbeck process, upper function, anomalous diffusion, the law of the iterated logarithm

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1. Introduction. An important field of application of the theory of random processes is concerned with modeling of diffusions (see, for example, [1]). In the present paper, we consider a time-varying Ornstein–Uhlenbeck velocity process. Assume that on a complete probability space $\{\Omega, \mathscr{F}, \mathbf{P}\}$ a scalar random process X_t, $t \geqslant 0$, is defined and governed by the nonautonomous linear stochastic differential equation

$$dX_t = -\delta_t X_t\, dt + \sigma_t\, dw_t, \qquad X_0 = 0, \tag{1}$$

where δ_t, $t \geqslant 0$, is a function characterizing the stability rate of the corresponding linear deterministic equation (that is, $\delta_t > 0$, $t \geqslant 0$, and $\int_0^t \delta_v\, dv \to \infty$, $t \to \infty$), σ_t, $t \geqslant 0$, is a piecewise continuous time function (the diffusion coefficient), and w_t, $t \geqslant 0$, is the standard Wiener process. In (1) no additional constraints on the coefficients are imposed, and so we can consider settings involving decaying ($\sigma_t \to 0$, $t \to \infty$), growing ($\sigma_t \to \infty$, $t \to \infty$), or constant ($\sigma_t \equiv \sigma$) perturbations, and also deal with stabilities of various types: exponential ($\delta_t \equiv \delta > 0$), subexponential ($\delta_t \to 0$, $t \to \infty$), and superexponential ($\delta_t \to \infty$, $t \to \infty$).

For the motion time $T \geqslant 0$, define the displacement process Y_T, $T \geqslant 0$, by the well-known formula (see, for example, [2, section 5.8])

$$Y_T = \int_0^T X_t\, dt \tag{2}$$

and the mean square displacement

$$D_T = \mathbf{E}\biggl(\int_0^T X_t\, dt\biggr)^2. \tag{3}$$

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Since model (1), (2) has proved useful in describing the diffusion behavior as an alternative to Brownian motion where displacement $Y_T^{(w)}$ is w_T, a question naturally arises regarding the comparison of D_T and $\mathbf{E}[Y_T^{(w)}]^2 = \mathbf{E}w_T^2 = T$ as $T \to \infty$. This analysis outlines normal diffusions (with $D_T \sim T$) and anomalous diffusions (with $D_T \nsim T$). Here and below, the symbol $\sim$ is used to denote the asymptotic equivalence of two scalar functions; thus $f_t \sim g_t$ if $0 < \lim_{t\to\infty}(f_t/g_t) < \infty$.

Definition 1 (see [3]). *Let*

$$d_1 = \liminf_{T\to\infty} \frac{D_T}{T} \quad \text{and} \quad d_2 = \limsup_{T\to\infty} \frac{D_T}{T}.$$

If $0 < d_1 \leqslant d_2 < \infty$, then the diffusion is called normal; otherwise it is anomalous. We refer to it as a subdiffusion for $d_2 = 0$ or a superdiffusion for $d_1 = \infty$.

It should be noted that a standard Ornstein–Uhlenbeck process ($\delta_t = \delta > 0$, $\sigma_t = \sigma \neq 0$ in (1)) generates a normal diffusion with $D_T \sim T$. It is clear that the mean square displacement D_T can be looked upon as an important statistical characteristic of the process $Y_T = \int_0^T X_t\,dt$, but still this quantity does not allow us to capture possible pathwise variations of Y_T as $T \to \infty$ depending on the parameters δ_t and σ_t. The notion of an upper function enables us to establish such a deterministic bound with probability 1.

Definition 2 (see [4]). *A deterministic function $h_T \geqslant 0$, $T \geqslant 0$, is called an upper function for a scalar process Z_T, $T \geqslant 0$, if there exists a nonrandom constant $\overline{c}$ such that, with probability* 1,

$$\limsup_{T\to\infty} \frac{Z_T}{h_T} < \overline{c} < \infty. \tag{4}$$

This definition is a modification of the definition of an upper function used, for example, in [5, section 6.6], where $\overline{c} = 1$, and a function h_T is assumed to be nondecreasing and such that $h_T \to \infty$ as $T \to \infty$. Following [4], [6], we are concerned with the variation of h_T in view of the above relation $\sim$, and hence we impose no constraints on the values of $\overline{c}$ or on the character of the h_T dynamics in (4).

The upper function can be used to estimate the order of variation of a random process in time. Indeed, condition (4) means that there exist a constant $\overline{c}_0 > 0$ and an a.s. finite time t_0 such that the inequality $Z_T \leqslant \overline{c}_0 h_T$ holds with probability 1 for any $T > t_0$. If (1) defines a standard Ornstein–Uhlenbeck process, then

$$Y_T = \int_0^T X_t\,dt = \delta^{-1}\int_0^T \sigma\,dw_t - \delta^{-1}X_T = \delta^{-1}(\sigma W_T - X_T),$$

and the upper function of the process $Z_T = |Y_T|$ can be derived from the known results: in view of the law of the iterated logarithm, for the Wiener process w_T we have $h_T^{(1)} \sim \sqrt{T\ln\ln T}$ (see, for example, [7]), and the upper function $h_T^{(2)}$ of the process $|X_T|$ is defined as $h_T^{(2)} \sim \sqrt{\ln T}$ (see [6]). So, in the case of constant coefficients, we have $h_T \sim \sqrt{T\ln\ln T}$, and so the probabilistic behavior of the displacement process Y_T turns out to be close to that of the Wiener process w_T, rather than in mean square ($D_T = \mathbf{E}Y_T^2 \sim \mathbf{E}w_T^2 = T$), in the stronger sense of bounding curves also, that is, in terms of upper functions. The study of the integrated Ornstein–Uhlenbeck process (2) is also motivated by various applications thereof, for example, in modeling population size [8] and integrated volatility [9], and also in statistics [10] (see the survey [3]). It is worth pointing out that previous studies in this direction were focused on statistical estimation [11] or on distributions of functionals (see, for example, [12]) under fairly restrictive assumptions on stationarity of processes or boundedness of their parameters. The purpose of the present paper is to find upper functions for the integrated process (2), as defined by (1) with time-varying coefficients, and examine the behavior of anomalous diffusions in terms of upper functions.

The paper is organized as follows. In section 2 we present various results on the form of upper functions for displacement processes depending on parameters of the stability rate and the diffusion coefficient of the linear stochastic differential equation. In section 3 we introduce the notion of anomalous diffusion from the viewpoint of upper functions and compare the results of diffusion classification by type on the basis of upper functions and using the mean square displacement; illustrative examples are also given in section 2.

2. The main results. The main results of the present paper are obtained under the basic assumption on variation of the stability rate in (1).

ASSUMPTION $\mathscr{D}$. *The stability rate δ_t is a monotone differentiable function, $t \geqslant 0$, and the function $\phi_t = \dot{\delta}_t/\delta_t^2$ (the dot is used to denote the time derivative) satisfies at least one of the two conditions*

$$\lim_{t\to\infty} \phi_t = \kappa, \quad \lim_{t\to\infty} \frac{1}{\phi_t} = \widetilde{\kappa} \qquad (\kappa, \widetilde{\kappa} \leqslant 0 \text{ are constants}), \tag{5}$$

and, if $\kappa = \widetilde{\kappa} = -1$, then $\left|\int_0^\infty \left(1/t - \delta_t\right) dt\right| < \infty$.

Note that the cases $\kappa, \widetilde{\kappa} > 0$ are not considered, because in these cases we have $\delta_t < 0$, $t \geqslant 0$; that is, in this setting it is clear that the above assumptions about δ_t (see (1)) are violated because of the unstable coefficient in the process dynamics equation.

One of the conditions in (5) was previously used in [13] in the study of stochastic linear regulator problems as it was necessary to preserve stability properties of the coefficient $\widetilde{\delta}_t = (1 + \phi_t)\delta_t$ in the equation of the transformed process $(1/\delta_t)X_t$. Assumption $\mathscr{D}$ in the above form is introduced for the first time in the present paper. Below, this assumption is used in the derivation of upper functions of the displacement process with adjusted velocity. Under Assumption $\mathscr{D}$ this process of adjusted displacement turns out to be equivalent in mean square to the initial process Y_T; we discuss this in detail below (see also Examples 3 and 4 in section 3). For cases where Assumption $\mathscr{D}$ is not satisfied, we suggest some possible methods of finding upper functions in Remark 1 at the end of the present section and provide Examples 1 and 2 for illustration.

In [3] it was shown that the mean square displacement can be written as the integrated variance of the process X_t, $t \geqslant 0$, as follows:

$$D_T = 2 \int_0^T \int_0^t \exp\left\{-\int_s^t \delta_v \, dv\right\} \mathbf{E} X_s^2 \, ds \, dt; \tag{6}$$

here

$$\mathbf{E} X_t^2 = \int_0^t \exp\left\{-2\int_s^t \delta_v \, dv\right\} \sigma_s^2 \, ds. \tag{7}$$

We now proceed similarly to the derivation of the upper function of the displacement in the case of a standard Ornstein–Uhlenbeck process (see the introduction) and obtain a closed-form representation; however, we use the adjusted velocity process $X_t^{(\psi)} = \psi_t X_t$ (here the deterministic function ψ_t tends to $\psi^* \neq 0$ as $t \to \infty$, where ψ^* is some constant) as a diffusion-generating process. Then the processes Y_T and $Y_T^{(\psi)}$ are characterized by mean square displacements of the same order, that is, $D_T \sim D_T^{(\psi)}$. More precisely,

$$Y_T^{(\psi)} = \int_0^T \psi_t X_t \, dt \quad \text{and} \quad D_T^{(\psi)} = 2\int_0^T \int_0^t \psi_s \psi_t \exp\biggl\{-\int_s^t \delta_v \, dv\biggr\} \mathbf{E} X_s^2 \, ds \, dt.$$

By the assumption on the function ψ_t, we have

$$0 < \lim_{T\to\infty} \frac{D_T^{(\psi)}}{D_T} < \infty \quad \text{and} \quad D_T \sim D_T^{(\psi)}.$$

We represent the process $Y_T^{(\psi)}$ as

$$Y_T^{(\psi)} = \int_0^T \psi_t X_t \, dt = \Psi_T X_T - \int_0^T \Psi_t \sigma_t \, dw_t, \tag{8}$$

where

$$\Psi_t = \int_0^t \exp\biggl\{\int_s^t \delta_v \, dv\biggr\} \psi_s \, ds.$$

Hence, the upper function of the process $Z_T = |Y_T^{(\psi)}|$ is determined by two components: the upper function $h_T^{(1)}$ for $Z_T^{(1)} = |\Psi_T X_T|$ and the upper function $h_T^{(2)}$ for $Z_T^{(2)} = \bigl|\int_0^T \Psi_t \sigma_t \, dw_t\bigr|$. In what follows, the process $Y_T^{(\psi)}$ is called the adjusted displacement or the displacement process for the adjusted velocity. The form of the adjustment function ϕ_t is taken depending on the values of the constants κ, $\widetilde{\kappa}$ from Assumption $\mathscr{D}$ (see (5)). First, we consider the case $-1 < \kappa \leqslant 0$ and set $\psi_t = 1 + \phi_t$. Then $\psi_t \to \psi^* = 1 + \kappa > 0$ as $t \to \infty$, $\Psi_T = -1/\delta_t$, and representation (8) for the adjusted displacement assumes the form

$$Y_T^{(\psi)} = -\frac{X_T}{\delta_T} + \int_0^T \frac{\sigma_t}{\delta_t} \, dw_t. \tag{9}$$

Here it is important to note that if the coefficients δ_t and σ_t are such that $Y_T^{(\psi)} \to Y_\infty^{(\psi)}$ with probability 1 as $T \to \infty$, where $Y_\infty^{(\psi)}$ is a random variable, then the estimate in terms of upper functions makes no sense, because in this setting one can take as a majorant any increasing function $h_T > 0$, $T \geqslant 0$, such that $h_T \to \infty$, $T \to \infty$. So, we first find out when such a case is possible.

LEMMA 1. *Let Assumption* $\mathscr{D}$ *be satisfied with constant* $-1 < \kappa \leqslant 0$, *let* $\int_0^\infty (\sigma_t^2/\delta_t^2) \, dt < \infty$, *and let the process* $Y_T^{(\psi)}$ *be defined in* (9). *Then* $D_\infty^{(\psi)} < \infty$, *and, moreover,* $Y_T^{(\psi)} \to Y_\infty^{(\psi)}$ *a.s. as* $T \to \infty$, *where* $Y_\infty^{(\psi)} = \int_0^\infty (\sigma_t/\delta_t) \, dw_t$.

Proof. Consider (9). It is known (see [14, section 5.4]) that for a Riemann-type integral (and in particular, for $Y_T^{(\psi)}$), the relation $Y_T^{(\psi)} \to Y_\infty^{(\psi)}$ a.s., $T \to \infty$, holds if

$\mathbf{E}[Y_T^{(\psi)}]^2$ is bounded for all $T \geqslant 0$. We have $\mathbf{E}[Y_T^{(\psi)}]^2 = D_T^{(\psi)}$, and hence it is required to examine the functions $D_T^{(\psi)}$ and D_T of the mean square displacement. From (9) we have $D_T^{(\psi)} \leqslant 2\mathbf{E}[X_T^2/\delta_T^2] + 2\int_0^\infty (\sigma_t^2/\delta_t^2)\,dt$. By the assumption, the second term is bounded. The process $\widetilde{X}_t = -X_t/\delta_t$ is given by the equation

$$d\widetilde{X}_t = -\widetilde{\delta}_t \widetilde{X}_t\,dt - \frac{\sigma_t}{\delta_t}\,dw_t, \qquad \widetilde{X}_0 = 0, \tag{10}$$

where $\widetilde{\delta}_t = (1+\phi_t)\delta_t$ is a stability rate because $\widetilde{\delta}_t \sim \delta_t$ and due to the condition $-1 < \kappa \leqslant 0$ of Assumption $\mathscr{D}$. Therefore, the expectation

$$\mathbf{E}\widetilde{X}_T^2 = \int_0^T \exp\biggl\{-2\int_t^T \widetilde{\delta}_v\,dv\biggr\}\frac{\sigma_t^2}{\delta_t^2}\,dt$$

is also bounded. Therefore, $D_\infty^{(\psi)} < \infty$ and $Y_T^{(\psi)} \to Y_\infty^{(\psi)}$ a.s. as $T \to \infty$. Next, we show that under the above assumptions $\widetilde{X}_T \to 0$ a.s., $T \to \infty$, and hence $Y_\infty^{(\psi)} = \int_0^\infty (\sigma_t/\delta_t)\,dw_t$. First, we note that if $\int_0^\infty (\sigma_t^2/\delta_t^2)\,dt < \infty$, then $\mathbf{E}\widetilde{X}_T^2 \to 0$, $T \to \infty$. The proof of this fact for bounded coefficients δ_t and σ_t can be found in [15], [16]; here we proceed analogously. Given a fixed arbitrarily small number $\varepsilon > 0$, for $\int_{t_0}^T (\sigma_t^2/\delta_t^2)\,dt < \varepsilon$, where $T > t_0(\varepsilon)$, we have

$$\begin{aligned}
\int_0^T \exp\biggl\{2\int_0^t \widetilde{\delta}_v\,dv\biggr\}\biggl(\frac{\sigma_t}{\delta_t}\biggr)^2 dt &= \exp\biggl\{2\int_0^T \widetilde{\delta}_v\,dv\biggr\}\int_0^T \biggl(\frac{\sigma_t}{\delta_t}\biggr)^2 dt \\
&\quad - 2\int_0^T \widetilde{\delta}_t \exp\biggl\{2\int_0^t \widetilde{\delta}_v\,dv\biggr\}\int_0^t \biggl(\frac{\sigma_t}{\delta_t}\biggr)^2 ds\,dt \\
&< -\exp\biggl\{2\int_0^T \widetilde{\delta}_v\,dv\biggr\}\int_{t_0}^T \biggl(\frac{\sigma_t}{\delta_t}\biggr)^2 dt + \varepsilon \exp\biggl\{2\int_0^T \widetilde{\delta}_v\,dv\biggr\} \\
&< \varepsilon \exp\biggl\{2\int_0^T \widetilde{\delta}_v\,dv\biggr\}.
\end{aligned}$$

As a result, $\mathbf{E}\widetilde{X}_T^2 \to 0$ as $T \to \infty$. Moreover, the process $\widetilde{X}_t$ asymptotically converges to zero, rather than just in the mean square, with probability 1 also. Consider (10) in the integral form

$$\widetilde{X}_T = -\int_0^T \widetilde{\delta}_t \widetilde{X}_t\,dt - \int_0^T \frac{\sigma_t}{\delta_t}\,dw_t.$$

Integration by parts shows that the expression

$$\begin{aligned}
&\mathbf{E}\biggl(\int_0^T \widetilde{\delta}_t \widetilde{X}_t\,dt\biggr)^2 \\
&\quad = 2\int_0^T \widetilde{\delta}_t \exp\biggl\{-\int_0^t \widetilde{\delta}_v\,dv\biggr\}\int_0^t \widetilde{\delta}_s \exp\biggl\{\int_0^s \widetilde{\delta}_v\,dv\biggr\}\mathbf{E}\widetilde{X}_s^2\,ds\,dt \\
&\quad < \int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt
\end{aligned}$$

is bounded. Therefore, $\int_0^T \widetilde{\delta}_t \widetilde{X}_t\,dt \to \int_0^\infty \widetilde{\delta}_t \widetilde{X}_t\,dt$ a.s., $T \to \infty$. Moreover, the integral $\int_0^\infty (\sigma_t/\delta_t)\,dw_t$ also exists, and hence $\widetilde{X}_T \to \widetilde{X}_\infty$ with probability 1, where $\widetilde{X}_\infty$ is some random variable. According to the above, $\widetilde{X}_T \to 0$ in mean square, and since the limit random variables should coincide a.s. (see [14, section 3.5]), we have $\widetilde{X}_\infty = 0$. Returning to (9), we see that the adjusted displacement process $Y_T^{(\psi)}$ converges to $\int_0^\infty (\sigma_t/\delta_t)\, dw_t$ a.s. as $T \to \infty$. The lemma is proved.

It should be pointed out that in the rest of the cases, we always have $D_T \to \infty$, $D_T^{(\psi)} \to \infty$ as $T \to \infty$ from condition (5) of Assumption $\mathscr{D}$ (that is, for $\kappa \leqslant -1$ and $-1 \leqslant \widetilde{\kappa} \leqslant 0$). Indeed, $\delta_t^{-1} \exp\{-\int_0^t \delta_v\, dv\} \to \infty$ as $t \to \infty$ if $-1 < \widetilde{\kappa} \leqslant 0$, and $\widehat{c}_1 \leqslant \delta_t^{-1} \exp\{-\int_0^t \delta_v\, dv\} \leqslant \widehat{c}_2$ if $\widetilde{\kappa} = -1$ by virtue of the condition $\left|\int_0^\infty (1/t - \delta_t)\, dt\right| < \infty$ (here $\widehat{c}_1, \widehat{c}_2 > 0$ are some constants). Hence, from (6), for the mean square displacement we have $D_T \geqslant c \int_0^T \delta_t \int_0^t \exp\{\int_0^t \delta_v\, dv\} \mathbf{E} X_s^2\, ds\, dt$ for $T > T_0(c)$ for some constant $c > 0$. By the definition of the stability rate δ_t, we have $\int_0^T \delta_t\, dt \to \infty$, $T \to \infty$, and hence we find also that $D_T \to \infty$, $T \to \infty$.

The next result gives the form of the upper function in the case $-1 < \kappa \leqslant 0$.

THEOREM 1. *Let Assumption $\mathscr{D}$ hold with constant $-1 < \kappa \leqslant 0$. Then, for the adjusted displacement process $Y_T^{(\psi)}$, as defined in* (9), *we have the following*:

(a) *If $\int_0^\infty (\sigma_t^2/\delta_t^2)\, dt < \infty$, then $Y_T^{(\psi)} \to Y_\infty^{(\psi)} = \int_0^\infty (\sigma_t/\delta_t)\, dw_t$ a.s. as $T \to \infty$*;

(b) *if $\int_0^T (\sigma_t^2/\delta_t^2)\, dt \to \infty$, $T \to \infty$, then the upper function of the process $Z_T = |Y_T^{(\psi)}|$ has the form*

$$h_T \sim \sqrt{\mathbf{E}\widetilde{X}_T^2 \ln \int_0^T \delta_t\, dt} + \sqrt{\int_0^T \frac{\sigma_t^2}{\delta_t^2}\, dt\, \ln\ln \int_0^T \frac{\sigma_t^2}{\delta_t^2}\, dt}, \tag{11}$$

where $\mathbf{E}\widetilde{X}_t^2 = \int_0^t \exp\{-2\int_s^t \widetilde{\delta}_v\, dv\}(\sigma_s^2/\delta_s^2)\, ds$, $\widetilde{\delta}_t = (1+\phi_t)\delta_t$, $\phi_t = \dot{\delta}_t/\delta_t^2$.

Proof. Case (a) is considered above in detail in the proof of Lemma 1. Regarding case (b) and representation (9), we see that the process $\widetilde{X}_t = -X_t/\delta_t$ satisfies (10). We have

$$\widetilde{X}_T = \exp\left\{-\int_0^T \widetilde{\delta}_v\, dv\right\} \int_0^T \exp\left\{\int_0^t \widetilde{\delta}_v\, dv\right\} \frac{\sigma_t}{\delta_t}\, dw_t,$$

and so, by the law of the iterated logarithm for stochastic integrals (see [17]), the upper function for the process $Z_T^{(1)} = |\widetilde{X}_T|$ reads as

$$h_T^{(1)} \sim \exp\left\{-\int_0^T \delta_v\, dv\right\} \sqrt{\ln\ln M_T}, \qquad M_T = \int_0^T \exp\left\{\int_0^t 2\widetilde{\delta}_v\, dv\right\} \frac{\sigma_t^2}{\delta_t^2}\, dt.$$

Taking into account the equality $\mathbf{E}\widetilde{X}_t^2 = \int_0^t \exp\{-2\int_s^t \widetilde{\delta}_v\, dv\}(\sigma_s^2/\delta_s^2)\, ds$, we define the upper function $h_T^{(1)} \sim \sqrt{\mathbf{E}\widetilde{X}_T^2 \ln\ln M_T}$. Moreover,

$$\limsup_{T\to\infty} \frac{\ln\ln M_T}{\ln \int_0^T \delta_t\, dt + \ln\ln \widehat{M}_T} < \infty,$$

where the upper function $h_T^{(2)} \sim \sqrt{\widehat{M}_T \ln\ln \widehat{M}_T}$ for $Z_T^{(2)} = \left|\int_0^T (\sigma_t/\delta_t)\, dw_t\right|$ can be obtained using the quadratic characteristic $\widehat{M}_T = \int_0^T (\sigma_t^2/\delta_t^2)\, dt$. Combining the above results, we get h_T of the form (11). This proves Theorem 1.

Next, we consider the case when Assumption $\mathscr{D}$ is satisfied with the constants $\kappa = \widetilde{\kappa} = -1$ (this also implies that $\lim_{t\to\infty}(t\delta_t) = 1$). In what follows, by g_T we denote any monotone function with the following properties:

$$g_T > 0, \quad T \geqslant 0; \qquad g_T \to \infty, \quad T \to \infty. \tag{12}$$

For the displacement process $Y_T = \int_0^T X_t\,dt$, we have

$$\begin{aligned} Y_T &= \int_0^T \exp\biggl\{-\int_0^t \delta_v\,dv\biggr\}\int_0^t \exp\biggl\{\int_0^s \delta_v\,dv\biggr\}\sigma_s\,dw_s \\ &= m_T \int_0^T \exp\biggl\{\int_0^t \delta_v\,dv\biggr\}\sigma_t\,dw_t - \int_0^T \widehat{m}_t\sigma_t\,dw_t, \end{aligned} \tag{13}$$

where $m_t = \int_0^t \exp\{-\int_0^s \delta_v\,dv\}\,ds$, $\widehat{m}_t = \exp\{\int_0^t \delta_v\,dv\}\int_0^t \exp\{-\int_0^s \delta_v\,dv\}\,ds$. Moreover, from the condition $\bigl|\int_0^\infty (1/t - \delta_t)\,dt\bigr| < \infty$ of Assumption $\mathscr{D}$ it easily follows that the functions $m_t/\ln t$, $\exp\{\int_0^t \delta_v\,dv\}/t$, $\widehat{m}_t/(t\ln t)$, are bounded away from zero and from above.

LEMMA 2. *Let Assumption $\mathscr{D}$ hold with the constants $\kappa = \widetilde{\kappa} = -1$, and let*

$$M_T = \int_0^T t^2\sigma_t^2\,dt, \qquad \widehat{M}_T = \int_0^T t^2\ln^2(t+1)\,\sigma_t^2\,dt. \tag{14}$$

Then the upper function for the process $Z_T = |Y_T|$ has the following form:

(a) *If $\widehat{M}_\infty < \infty$, then $h_T \sim g_T \ln T$, where the function g_T is defined in* (12);

(b) *if $M_\infty < \infty$ and $\widehat{M}_T \to \infty$, $T \to \infty$, then $h_T \sim \ln T\sqrt{\ln\ln\ln T}$*;

(c) *if $M_T \to \infty$, $T \to \infty$, then $h_T \sim \ln T\sqrt{M_T \ln\ln M_T} + \sqrt{\widehat{M}_T \ln\ln \widehat{M}_T}$.*

Proof. Representation (13) can be put in the form

$$Y_T = m_T I_T - \widehat{I}_T,$$

where the stochastic integrals

$$I_T = \int_0^T \exp\biggl\{\int_0^t \delta_v\,dv\biggr\}\sigma_t\,dw_t \quad\text{and}\quad \widehat{I}_T = \int_0^T \widehat{m}_t\sigma_t\,dw_t$$

have the quadratic characteristics $\langle I_T\rangle$, $\langle \widehat{I}_T\rangle$, and moreover,

$$\begin{aligned} 0 < \liminf_{T\to\infty}\frac{\langle I_T\rangle}{M_T} \leqslant \limsup_{T\to\infty}\frac{\langle I_T\rangle}{M_T} < \infty, \\ 0 < \liminf_{T\to\infty}\frac{\langle \widehat{I}_T\rangle}{\widehat{M}_T} \leqslant \limsup_{T\to\infty}\frac{\langle \widehat{I}_T\rangle}{\widehat{M}_T} < \infty, \end{aligned}$$

where M_T and $\widehat{M}_T$ are defined in (14). Then, by the law of the iterated logarithm, for the stochastic integrals [17], we see that $h_T^{(1)} \sim \sqrt{M_T\ln\ln M_T}$ is the upper function for the process $|I_T|$ as $M_T \to \infty$, and $h_T^{(2)} \sim \sqrt{\widehat{M}_T\ln\ln\widehat{M}_T}$ is the upper function of the process $|\widehat{I}_T|$ as $\widehat{M}_T \to \infty$, $T \to \infty$. Further, if $M_\infty < \infty$, then $I_T \to I_\infty$, and $\widehat{I}_T \to \widehat{I}_\infty$ in the case $\widehat{M}_\infty < \infty$. Using these relations, we next obtain an upper function h_T depending on the asymptotic behavior of the quadratic characteristics. It is clear that a random variable with finite moments is majorized by any function $g_T > 0$ with properties (12). We have $M_T \leqslant \widehat{M}_T < \infty$ for $\widehat{M}_\infty < \infty$, and hence we have the upper function of the form $h_T \sim g_T \ln T$ for the process $Z_T = |Y_T|$. If $M_\infty < \infty$ and $\widehat{M}_T \to \infty$, $T \to \infty$, then, since $\widehat{M}_T \leqslant \ln^2(T+1)M_T$,

$$\limsup_{T\to\infty} \frac{h_T^{(2)}}{\ln T\sqrt{\ln\ln\ln T}} < \infty$$

for $h_T^{(2)}$. Now the resulting upper function reads as $h_T \sim \ln T\sqrt{\ln\ln\ln T}$. Finally, $h_T \sim h_T^{(1)} + h_T^{(2)}$ in the case $M_T \to \infty$, $T \to \infty$. Lemma 2 is proved.

We now analyze the situation when Assumption $\mathscr{D}$ holds with constant $-1 < \widetilde{\kappa} \leqslant 0$. Setting $\psi_t = 1 - t\delta_t$ in representation (8), we have $\Psi_t = t$, $\psi_t \to \psi^* = 1 - \widetilde{\kappa}$ as $t \to \infty$, and so (8) assumes the form

$$Y_T^{(\psi)} = TX_T - \int_0^T t\sigma_t\, dw_t. \tag{15}$$

THEOREM 2. *Let Assumption $\mathscr{D}$ hold with constant $-1 < \widetilde{\kappa} \leqslant 0$. Then the upper function for the process $Z_T = |Y_T^{(\psi)}|$, as defined by* (15), *has the following form*:

(a) *If $\int_0^\infty t^2\sigma_t^2\, dt < \infty$, then $h_T \sim Tg_T\sqrt{\mathbf{E}X_T^2}$*;

(b) *if $\int_0^\infty \exp\{2\int_0^t \delta_v\, dv\}\sigma_t^2\, dt < \infty$ and $\int_0^T t^2\sigma_t^2\, dt \to \infty$, $T \to \infty$, then*

$$h_T \sim Tg_T\sqrt{\mathbf{E}X_T^2} + \sqrt{\int_0^T t^2\sigma_t^2\, dt\, \ln\ln \int_0^T t^2\sigma_t^2\, dt};$$

(c) *if $\int_0^T \exp\{2\int_0^t \delta_v\, dv\}\sigma_t^2\, dt \to \infty$, $T \to \infty$, then*

$$h_T \sim T\sqrt{\mathbf{E}X_T^2\left(\ln \int_0^T \delta_t\, dt + \ln\ln \int_0^T \sigma_t^2\, dt\right)} + \sqrt{\int_0^T t^2\sigma_t^2\, dt\, \ln\ln \int_0^T t^2\sigma_t^2\, dt}.$$

Here, g_T is a monotone function satisfying (12), *and $\mathbf{E}X_T^2$ is calculated per* (7).

Proof. Regarding representation (15), it should be noted that the equation for the process $\widetilde{X}_t = tX_t$ (see (10)) contains the unstable coefficient $\widetilde{\delta}_t = (1 - 1/(t\delta_t))\delta_t < 0$, which results in growing upper functions. For example, in [18] it was shown that, in the case of constant coefficients in (10), the upper function grows exponentially.

In assertion (a) of Theorem 2, the condition $\int_0^\infty t^2\sigma_t^2\, dt < \infty$ also implies that $\int_0^T \exp\{2\int_0^t \delta_v\, dv\}\sigma_t^2\, dt < \infty$ and $\mathbf{E}X_T^2 \sim \exp\{-2\int_0^T \delta_v\, dv\}$, and it follows from here that

$$I_T = \int_0^T \exp\left\{\int_0^t \delta_v\, dv\right\}\sigma_t\, dw_t \xrightarrow{\text{a.s.}} I_\infty = \int_0^\infty \exp\left\{\int_0^t \delta_v\, dv\right\}\sigma_t\, dw_t,$$

and hence, as an upper function for the process

$$Z_T^{(1)} = |TX_T| = \left|T\exp\left\{-\int_0^T \delta_v\, dv\right\}I_T\right|$$

one can take the function $h_T^{(1)} \sim T g_T \sqrt{\mathbf{E}X_T^2}$ (g_T is defined in (12)), which is also a majorant of the process $Z_T^{(2)} = |\widehat{I}_T|$, where

$$\widehat{I}_T = \int_0^T t\sigma_t \, dw_t \to \widehat{I}_\infty = \int_0^\infty t\sigma_t \, dw_t.$$

The function h_T reads as $h_T \sim h_T^{(1)} \to \infty$, $T \to \infty$, since the function $T \exp\{-\int_0^T \delta_v \, dv\}$ is bounded away from zero by the condition $-1 < \widetilde{\kappa} \leqslant 0$ of Assumption $\mathscr{D}$.

Assertion (b) of the theorem is clear if one looks at the form of the upper function $h_T^{(2)}$ for the process $Z_T^{(2)} = |\widehat{I}_T|$ according to the law of the iterated logarithm and combines this estimate with the above upper function $h_T^{(1)}$.

If $\int_0^T \exp\{2\int_0^t \delta_v \, dv\}\sigma_t^2 \, dt \to \infty$, $T \to \infty$, then the derivation of the upper function $h_T^{(1)}$ for $Z_T^{(1)} = |TX_T|$ is similar to that given in the proof of Theorem 1; that is, $h_T^{(1)} \sim T\sqrt{\mathbf{E}X_T^2(\ln\int_0^T \delta_t \, dt + \ln\ln\int_0^T \sigma_t^2 \, dt)}$. This in combination with $h_T^{(2)} \sim \sqrt{\int_0^T t^2\sigma_t^2 \, dt \, \ln\ln\int_0^T t^2\sigma_t^2 \, dt}$ gives the representation in assertion (c) of Theorem 2. Theorem 2 is proved.

Remark 1. The form of the upper functions was obtained above for the adjusted displacement $Y_T^{(\psi)}$ (see (8)), which is equivalent under the hypothesis of Assumption $\mathscr{D}$ to the original process Y_T in the mean square sense; that is, $D_T^{(\psi)} \sim D_T$. In the absence of Assumption $\mathscr{D}$ the form of upper functions can also be derived using the above arguments.

First, if one manages to replace the equivalence requirement $D_T^{(\psi)} \sim D_T$ by the weaker condition of mean square comparability of the processes $Y_T^{(\psi)}$ and Y_T, that is,

$$0 < \liminf_{T\to\infty} \frac{D_T^{(\psi)}}{D_T} \leqslant \limsup_{T\to\infty} \frac{D_T^{(\psi)}}{D_T} < \infty,$$

then the above results on upper functions for the processes $Y_T^{(\psi)}$ can be used. The processes are comparable in the above sense, when, in particular, instead of (5), one has

$$\kappa_2 \leqslant \liminf_{T\to\infty} \phi_t \leqslant \limsup_{T\to\infty} \phi_t \leqslant \kappa_1 \leqslant 0,$$

where $\kappa_1, \kappa_2 \leqslant 0$ are constants such that $1 + 2\kappa_2 + \kappa_1^2 > 0$.

Second, regardless of the behavior of ϕ_t, upper functions can be derived by using (8) with $\psi_t \equiv 1$, that is, by direct integration of the velocity process (see also representation (13), as obtained for Lemma 2). Consider the function B_t defined by

$$B_t = -\int_t^\infty \exp\left\{-\int_0^s \delta_v \, dv\right\} ds \quad \text{if} \quad \int_0^\infty \exp\left\{-\int_0^t \delta_v \, dv\right\} dt < \infty \tag{16}$$

and

$$B_t = \int_0^t \exp\left\{-\int_0^s \delta_v \, dv\right\} ds \quad \text{if} \quad \int_0^t \exp\left\{-\int_0^s \delta_v \, dv\right\} ds \to \infty, \ t \to \infty. \tag{17}$$

Then the displacement process Y_T can be put in the form

$$\begin{aligned} Y_T &= \int_0^T X_t\,dt \\ &= B_T \int_0^T \exp\biggl\{\int_0^t \delta_v\,dv\biggr\}\sigma_t\,dw_t - \int_0^T B_t \exp\biggl\{\int_0^t \delta_v\,dv\biggr\}\sigma_t\,dw_t. \end{aligned} \tag{18}$$

Below we present a result on upper functions of the displacement process. This result is based on the analysis of (18), which repeats steps in the proof of Lemma 2. The next lemma extends Lemma 2 to the case of an arbitrary velocity process X_t with dynamics (1).

LEMMA 3. *Let*

$$M_T^{(1)} = \int_0^T \exp\biggl\{2\int_0^t \delta_v\,dv\biggr\}\sigma_t^2\,dt, \quad M_T^{(2)} = \int_0^T B_t^2 \exp\biggl\{2\int_0^t \delta_v\,dv\biggr\}\sigma_t^2\,dt, \tag{19}$$

where the function B_t is defined in (16), (17). *Then the upper function h_T of the process $Z_T = |Y_T|$ is of the following form*:

(1) *If $M_\infty^{(1)} < \infty$, $M_\infty^{(2)} < \infty$, then*

$$\text{(1.a)} \quad Y_T \to Y_\infty = -\int_0^\infty B_t \exp\biggl\{\int_0^t \delta_v\,dv\biggr\}\sigma_t\,dw_t \quad a.s., \quad T\to\infty,$$

$$\textit{in the case} \quad \int_0^\infty \exp\biggl\{-\int_0^t \delta_v\,dv\biggr\}\,dt < \infty;$$

$$\text{(1.b)} \quad h_T \sim g_T|B_T|$$

$$\textit{in the case} \quad \int_0^t \exp\biggl\{-\int_0^s \delta_v\,dv\biggr\}\,ds \to \infty, \quad t\to\infty;$$

(2) *if $M_\infty^{(1)} < \infty$ and $M_T^{(2)} \to \infty$, $T\to\infty$, then*

$$h_T \sim |B_T|\sqrt{\ln\ln|B_T|};$$

(3) *if $M_T^{(1)} \to \infty$, $T\to\infty$, and $M_\infty^{(2)} < \infty$, then*

$$h_T \sim |B_T|\sqrt{M_T^{(1)}\ln\ln M_T^{(1)}} + g_T;$$

(4) *if $M_T^{(1)} \to \infty$, $M_T^{(2)} \to \infty$, $T\to\infty$, then*

$$h_T \sim |B_T|\sqrt{M_T^{(1)}\ln\ln M_T^{(1)}} + \sqrt{M_T^{(2)}\ln\ln M_T^{(2)}};$$

here g_T is a function with properties (12).

The following examples illustrate the above approaches to the derivation of upper functions in cases when Assumption $\mathscr{D}$ is not met.

Example 1. Consider the family of stability rate functions

$$\delta_t^{(k)} = \frac{1}{k(t+1) + (k/2)\sin(2t)}, \qquad 0 < k < \frac{1}{4}.$$

Let the velocity process $X_t^{(k)}$, $t \geqslant 0$, be governed by equation (1) for $\delta_t = \delta_t^{(k)}$ and an arbitrary diffusion coefficient σ_t. The displacement process $Y_t^{(k)}$ and the mean square displacement $D_T^{(k)}$ are defined by equations (2) and (3), respectively. For the above $\delta_t^{(k)}$, it is clear that the function $\phi_t^{(k)} = 2k(-1+\sin^2 t)$ does not satisfy relations (5) of Assumption $\mathscr{D}$. The adjusted displacement is defined by (9), and the inequality $(1-4k)D_T^{(k)} \leqslant D_T^{(\psi)} \leqslant D_T^{(k)}$ holds for it. Hence, in the mean square sense, the processes $Y_t^{(k)}$ and $Y_t^{(\psi)}$ turn out to be comparable, and so the upper function can be found from the previous results. Note that just the stability of the coefficient $\widetilde{\delta}_t = \delta_t(1+\phi_t)$ was required in Theorem 1 to determine the form of the upper function, that is, the property $\widetilde{\delta}_t > 0$, $t \geqslant 0$. In this case, this property also holds for $\widetilde{\delta}_t^{(k)} = \delta_t^{(k)}(1+\phi_t^{(k)})$, because $\delta_t^{(k)} > 0$ and $0 < 1-2k \leqslant 1+\phi_t^{(k)} \leqslant 1$. Then assertions (a) and (b) of Theorem 1 remain valid if δ_t is replaced by $\delta_t^{(k)}$, $\widetilde{\delta}_t$ is replaced by $\widetilde{\delta}_t^{(k)}$, and, instead of the functions ϕ_t and $\mathbf{E}\widetilde{X}_T^2$, one puts, respectively, the functions $\phi_t^{(k)}$ and $\mathbf{E}(\widetilde{X}_T^{(k)})^2 = \mathbf{E}(X_T^{(k)})^2/(\delta_t^{(k)})^2$. Using the formulas

$$\delta_t^{(k)} \sim \frac{1}{t}, \quad \int_0^t \delta_v^{(k)}\,dv \sim \int_0^t \frac{dv}{v+1}, \quad \mathbf{E}(X_T^{(k)})^2 \sim \frac{1}{(T+1)^{2k}} \int_0^T (t+1)^{2k}\sigma_t^2\,dt$$

to simplify the above expressions, we see that the upper function has the form

$$h_T \sim h_T^{(1)} + h_T^{(2)},$$

where

$$h_T^{(1)} = T^{1-k}\sqrt{\ln\ln T}\sqrt{\int_0^T t^{2k}\sigma_t^2\,dt}, \quad h_T^{(2)} = \sqrt{\int_0^T t^2\sigma_t^2\,dt\,\ln\ln\int_0^T t^2\sigma_t^2\,dt},$$

if $\int_0^T t^2\sigma_t^2\,dt \to \infty$ as $T\to\infty$, and

$$Y_T^{(\psi)} \to Y_\infty^{(\psi)} = \int_0^\infty \frac{\sigma_t}{\delta_t^{(k)}}\,dw_t \quad \text{a.s.}$$

if $\int_0^\infty t^2\sigma_t^2\,dt < \infty$.

Example 2. Let

$$\delta_t = \frac{1}{t+1} + \frac{a}{(t+1)\ln(t+b)},$$

where a, b are real numbers, $a \neq 0$, $b > 1$. For this family of functions, despite the fact that the constants κ and $\widetilde{\kappa}$ in (5) are equal to -1, we have

$$\left|\int_0^t \left(\frac{1}{s+1} - \delta_s\right) ds\right| \to \infty, \qquad s\to\infty;$$

that is, Assumption $\mathscr{D}$ does not hold. We use Lemma 3 to find the form of the upper function. In (19) the function B_t reads as

$$B_t \sim \int \frac{1}{(t+1)\ln^a(t+b)}\,dt,$$

and hence $B_t \sim \ln\ln(t+b)$ for $a=1$ and $|B_t| \sim \ln^{1-a}(t+b)$ for $a \neq 1$. In the above example,

$$M_T^{(1)} \sim \int_0^T t^2 \ln^{2a}(t+b)\,\sigma_t^2\,dt.$$

For the second quadratic characteristic $M_T^{(2)}$, we have

$$M_T^{(2)} \sim \int_0^T t^2 \ln^2(t+b)\,\sigma_t^2\,dt \quad \text{if} \quad a \neq 1$$

and

$$M_T^{(2)} \sim \int_0^T t^2 \ln^2(t+b)\ln^2\ln(t+b)\,\sigma_t^2\,dt \quad \text{if} \quad a = 1.$$

We also note that if $t \to \infty$, then $B_t \to \infty$ in the case $a \leqslant 1$, and $B_t \to 0$ in the case $a > 1$, which enables us to find out which of the cases of assertions (1)–(4) in Lemma 3 are possible for various values of the parameter a. For $a \leqslant 1$, assertion (1.b) holds in the case $M_\infty^{(2)} < \infty$, while assertion (2) is true when $M_\infty^{(1)} < \infty$ and $M_T^{(2)} \to \infty$, $T \to \infty$. If, further, $a > 1$, then the relation from assertion (1.a) holds if $M_\infty^{(1)} < \infty$; the upper function of assertion (3) ensues if $M_T^{(1)} \to \infty$, $T \to \infty$, $M_\infty^{(2)} < \infty$. The result of assertion (4) appears in the case $M_T^{(1)} \to \infty$, $M_T^{(2)} \to \infty$, $T \to \infty$, with no constraints on the parameter a.

3. On the matching of the diffusion types detected from the mean square displacement behavior and upper functions. The present section is concerned with the issue of matching the diffusion types that are detected from the mean square displacement (see Definition 1) and on the basis of upper functions. Extension of the approach towards determination of the diffusion type by comparing its characteristics with the known characteristics of the normal diffusion (in the case under consideration, with the upper function $h_T \sim \sqrt{T\ln\ln T}$) leads us to the following definition.

DEFINITION 3. *Assume that we know h_T, which is the upper function of the displacement process Y_T or the adjusted displacement process $Y_T^{(\psi)}$ (see Theorems 1 and 2 and Lemmas 1 and 2). Let*

$$\overline{d}_1 = \liminf_{t\to\infty} \frac{h_T}{\sqrt{T\ln\ln T}} \quad \textit{and} \quad \overline{d}_2 = \limsup_{t\to\infty} \frac{h_T}{\sqrt{T\ln\ln T}}.$$

If $0 < \overline{d}_1 \leqslant \overline{d}_2 < \infty$, then the diffusion is called normal from the point of view of the upper function; otherwise, the diffusion is anomalous; we refer to it as a subdiffusion for $\overline{d}_2 = 0$ or a superdiffusion for $\overline{d}_1 = \infty$.

As pointed out above, in the case $Y_T \to Y_\infty$, $T \to \infty$, where Y_∞ is a random variable, one can take any positive unboundedly increasing function $h_T \sim g_T$ as an upper function h_T (see (12)). Hence, we have a subdiffusion by Definition 3. The next example shows that the results of detection of the diffusion type from the mean square displacement and on the basis of upper functions may differ from each other.

Example 3. Consider the displacement and adjusted displacement processes in the cases when the velocity equation (1) contains equal diffusion coefficients σ_t, but the coefficients (stability rates) $\delta_t^{(i)}$ are different: $\delta_t^{(1)} \sim \delta_t^{(2)} \sim \delta_t^{(3)} \sim 1/t$. More precisely, $\delta_t^{(i)} = k_i/t$, where k_i are constants: $k_1 > 1$, $k_2 = 1$, $0 < k_3 < 1$. Hence, we see that the constants in Assumption $\mathscr{D}$ are as follows:

$$-1 < \kappa_1 = -k_1^{-1} < 0, \quad \kappa_2 = -k_2 = -1,$$
$$\kappa_3 = -k_3^{-1} < -1, \quad -1 < \widetilde{\kappa}_3 = -k_3 < 0.$$

Assume also that

$$\int_0^\infty \frac{\sigma_t^2}{(\delta_t^{(i)})^2}\, dt \sim \int_0^\infty t^2 \sigma_t^2\, dt < \infty, \qquad i = 1, 2, 3.$$

Then for $\delta_t^{(1)} = k_1/t$, by assertion (a) of Theorem 1, the adjusted displacement $Y_T^{(\psi)}$ converges to $Y_\infty^{(\psi)} = (1/k_1) \int_0^\infty t\sigma_t\, dw_t$, $D_\infty^{(\psi)} < \infty$, and we have the limit case of subdiffusion. For $\delta_t^{(2)} = 1/t$, using Lemma 2 (assertion (a) or (b)), we find that

$$\limsup_{T\to\infty} \frac{h_T}{\ln T \sqrt{\ln\ln\ln T}} < \infty$$

and

$$D_T^{(\psi)} \sim D_T = \int_0^T \frac{1}{t} \int_0^t \frac{s}{s^2} \int_0^s \tau^2 \sigma_\tau^2\, d\tau\, ds\, dt \sim \ln^2 T,$$

which also corresponds to a subdiffusion. If $\delta_t^{(3)} = k_3/t$ and $-1 < \widetilde{\kappa}_3 = -k_3 < 0$, then by assertion (a) of Theorem 2,

$$h_T \sim T g_T \exp\biggl\{-\int_0^T \frac{k_3}{t}\, dt\biggr\}, \quad \text{that is,} \quad h_T \sim T^{1-k_3} g_T,$$

and $D_T \sim T^{2(1-k_3)}$. The analysis of the dynamics of D_T shows that we have a normal diffusion for $k_2 = 1/2$, a superdiffusion for $0 < k_3 < 1/2$, and a subdiffusion for $1/2 < k_3 < 1$. Taking into account the form of the upper functions, we detect a subdiffusion for $1/2 < k_3 < 1$ and a superdiffusion for $0 < k_3 < 1/2$. However, for $k_3 = 1/2$ a subdiffusion is also detected: $h_T \sim \sqrt{T} g_T$ and $h_T/\sqrt{T \ln\ln T} \to 0$, $T \to \infty$, if a function g_T grows slower than $\sqrt{\ln\ln T}$. So, in this case we have a disagreement with the conclusion about the diffusion type based on the mean square displacement — this is because the representation for the upper function involves an arbitrary increasing function g_T.

Let us now proceed with a more detailed analysis of situations of mutual determination of the diffusion types by using the mean square displacement (see Definition 1) and upper functions (see Definition 3). In [3], the diffusion type was detected by comparing the variance $\mathbf{E}X_t^2$ of the process X_t, $t \geqslant 0$, and the stability rate function δ_t. More precisely, analyzing the limits of the ratio $\mathbf{E}X_t^2/\delta_t$, $t \to \infty$, the following result was proved.

THEOREM 3 (see [3]). *Let*

$$d_1^* = \liminf_{t\to\infty} \frac{\mathbf{E}X_t^2}{\delta_t} \quad \textit{and} \quad d_2^* = \limsup_{t\to\infty} \frac{\mathbf{E}X_t^2}{\delta_t},$$

where $\mathbf{E}X_t^2$ *is defined in* (7). *Then the following types of diffusions take place*:
(1) *a normal diffusion for* $0 < d_1^* \leqslant d_2^* < \infty$,
(2) *a subdiffusion for* $d_2^* = 0$,
(3) *a superdiffusion for* $d_1^* = \infty$.

Remark 2. If Assumption $\mathscr{D}$ holds with $-2 < \kappa \leqslant 0$, then the conditions of assertions (1)–(3) of Theorem 3 can be replaced, respectively, by the following requirements based on the limit behavior as $t \to \infty$ of the relation σ_t^2/δ_t^2:
(1) $\sigma_t^2/\delta_t^2 \to c^*$ for some constant $c^* > 0$,
(2) $\sigma_t^2/\delta_t^2 \to 0$,
(3) $\sigma_t^2/\delta_t^2 \to \infty$.

The relations from Theorem 3 appear below when figuring out agreement/disagreement between diffusion types obtained from the analysis of mean square displacements and upper functions.

THEOREM 4. *Assume that Assumption* $\mathscr{D}$ *holds with* $-1 < \kappa \leqslant 0$ *and* $\liminf_{t\to\infty}(\dot{\delta}_t t/\delta_t) \geqslant 0$ *with* $\kappa = 0$. *Then the types of diffusions, as detected from the mean square displacement dynamics and the upper functions, coincide and can be determined from assertions* (1)–(3) *of Theorem* 3 (*the quantities* d_1^* *and* d_2^* *are defined in Theorem* 3) *as follows*:
(1) *a normal diffusion for* $0 < d_1^* \leqslant d_2^* < \infty$,
(2) *a subdiffusion for* $d_2^* = 0$,
(3) *a superdiffusion for* $d_1^* = \infty$.

Proof. In the case $-1 < \kappa \leqslant 0$, the form of the upper function is given in assertion (b) of Theorem 1:

$$h_T \sim h_T^{(1)} + h_T^{(2)}, \tag{20}$$

where

$$h_T^{(1)} = \sqrt{\mathbf{E}\widetilde{X}_T^2 \ln \int_0^T \delta_t\,dt}, \qquad h_T^{(2)} = \sqrt{\int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt\,\ln\ln \int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt}. \tag{21}$$

Consider $\mathbf{E}\widetilde{X}_T^2 = \mathbf{E}X_T^2/\delta_T^2$,

$$\mathbf{E}\widetilde{X}_T^2 = \int_0^t \exp\biggl\{-2\int_s^t \widetilde{\delta}_v\,dv\biggr\}\frac{\sigma_s^2}{\delta_s^2}\,ds = \int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt - 2\int_0^T \widetilde{\delta}_t \mathbf{E}\widetilde{X}_t^2\,dt.$$

Hence, for $\int_0^T (\sigma_t^2/\delta_t^2)\,dt$, we have

$$\begin{gathered} \int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt = \mathbf{E}\widetilde{X}_T^2 + 2\int_0^T \widetilde{\delta}_t \mathbf{E}\widetilde{X}_t^2\,dt, \\ \text{where}\quad \widetilde{\delta}_t = (1+\phi_t)\delta_t, \quad \frac{\widetilde{\delta}_t}{\delta_t} \to 1+\kappa > 0, \quad t \to \infty. \end{gathered} \tag{22}$$

Let us begin the proof of the theorem with assertion (3), when $\delta_t \mathbf{E}\widetilde{X}_t^2 \to \infty$, $t \to \infty$, and, correspondingly, $\int_0^T \widetilde{\delta}_t \mathbf{E}\widetilde{X}_t^2\,dt \to \infty$, $T \to \infty$. Using (22), this gives

$$\frac{1}{T}\int_0^T \frac{\sigma_t^2}{\delta_t^2}\,dt \geqslant \frac{1}{T}\int_0^T \widetilde{\delta}_t \mathbf{E}\widetilde{X}_t^2\,dt \to \infty, \qquad T \to \infty.$$

Now an appeal to representation (20), (21) shows that

$$\frac{h_T}{\sqrt{T\ln\ln T}} \geqslant \frac{h_T^{(2)}}{\sqrt{T\ln\ln T}} \to \infty, \qquad T\to\infty,$$

and so this is the case of a superdiffusion.

In assertion (2), vice versa, $\delta_t\mathbf{E}\widetilde{X}_t^2 \to 0$, $t\to\infty$. In this case we have $\mathbf{E}\widetilde{X}_T^2/T \to 0$, $T\to\infty$. Indeed, $\mathbf{E}\widetilde{X}_T^2/T = \delta_T\mathbf{E}\widetilde{X}_T^2/(T\delta_T)$, and, moreover, $\liminf_{T\to\infty}(T\delta_T) > 0$ by the condition $-1<\kappa\leqslant 0$ of Assumption $\mathscr{D}$ (more precisely, $T\delta_T\to\infty$ as $T\to\infty$ if $\kappa=0$, and $T\delta_T \to -1/\kappa$ as $T\to\infty$ if $-1<\kappa<0$). Hence $T^{-1}\int_0^T(\sigma_t^2/\delta_t^2)\,dt \to 0$, $T\to\infty$, and so, in (20), (21),

$$\frac{h_T^{(2)}}{\sqrt{T\ln\ln T}} \to 0, \qquad T\to\infty.$$

Next, we consider the ratio

$$\frac{h_T^{(1)}}{\sqrt{T\ln\ln T}} = \sqrt{\delta_T\mathbf{E}\widetilde{X}_T^2 H_T}, \qquad \text{where} \quad H_T = \frac{\ln\int_0^T\delta_t\,dt}{\delta_T T\ln\ln T}. \tag{23}$$

To evaluate the limit on the right of (23), we argue as follows. If $-1<\kappa<0$, then $\delta_t\sim 1/t$, $\ln\int_0^T\delta_t\,dt \sim \ln\ln T$, and so $H_T\to c$ as $T\to\infty$ for some constant $c>0$. For $\kappa=0$, using L'Hôpital's rule and since $\liminf_{t\to\infty}(\dot\delta_t t/\delta_t)\geqslant 0$, we have $\lim_{T\to\infty}H_T \leqslant \lim_{T\to\infty}1/\bigl\{(\dot\delta_T T/\delta_T+1)\int_0^T\delta_t\,dt\bigr\} = 0$. Hence, in both cases $\delta_T\mathbf{E}\widetilde{X}_T^2 H_T\to 0$, $T\to\infty$, and in (23),

$$\frac{h_T^{(1)}}{\sqrt{T\ln\ln T}} \to 0, \qquad T\to\infty,$$

which, in combination with the convergence $h_T^{(2)}/\sqrt{T\ln\ln T}\to 0$ already established, gives a subdiffusion from upper functions.

Under the hypotheses of assertion (1), the integral $\int_0^T(\sigma_t^2/\delta_t^2)\,dt$ is estimated with the help of the inequalities $\widetilde{d}_1 \leqslant \delta_t\mathbf{E}\widetilde{X}_t^2 \leqslant \widetilde{d}_2$, which hold for some positive constants $\widetilde{d}_1$, $\widetilde{d}_2$. As a result, we get

$$\frac{\widetilde{d}_1}{\delta_T T} + \frac{2\widetilde{d}_1}{T}\int_0^T\frac{\widetilde{\delta}_t}{\delta_t}\,dt \leqslant \frac{1}{T}\int_0^T\frac{\sigma_t^2}{\delta_t^2}\,dt \leqslant \frac{\widetilde{d}_2}{\delta_T T} + \frac{2\widetilde{d}_2}{T}\int_0^T\frac{\widetilde{\delta}_t}{\delta_t}\,dt,$$

and, moreover, since by the above, $\liminf_{T\to\infty}(T\delta_T)>0$ and $\widetilde{\delta}_t/\delta_t\to 1+\kappa>0$, $t\to\infty$, it can be concluded that

$$\widehat{d}_1 \leqslant \frac{1}{T}\int_0^T\frac{\sigma_t^2}{\delta_t^2}\,dt \leqslant \widehat{d}_2 \quad \text{for some constants} \ \ \widehat{d}_1,\widehat{d}_2>0. \tag{24}$$

From (24) it follows that

$$0 < \liminf_{T\to\infty}\frac{h_T^{(2)}}{\sqrt{T\ln\ln T}} \leqslant \limsup_{T\to\infty}\frac{h_T^{(2)}}{\sqrt{T\ln\ln T}} < \infty. \tag{25}$$

In the analysis of $h_T^{(1)}/\sqrt{T\ln\ln T}$, we put $h_T^{(1)}/\sqrt{T\ln\ln T} = \sqrt{\delta_T\mathbf{E}\widetilde{X}_T^2 H_T}$ (see (23)), as in the proof of assertion (2), and show that $\lim_{T\to\infty}H_T = 0$ if $\kappa=0$, and

$\lim_{T\to\infty} H_T > 0$ if $-1 < \kappa < 0$. Hence $\limsup_{T\to\infty}(h_T^{(1)}/\sqrt{T\ln\ln T}) < \infty$, which, in view of (20) and (25), enables one to classify the diffusion as normal from the point of view of upper functions. Theorem 4 is proved.

The arguments given below pertain to the agreement of diffusion types in the case $\kappa = -1$. Assume that the relation of assertion (2) in Theorem 3 is satisfied; that is, the process is a subdiffusion from the point of view of the mean square displacement dynamics. Therefore, $t\mathbf{E}X_t^2 \to 0$, $t \to \infty$, and $M_T/T \to 0$, $T \to \infty$, where $M_T = \int_0^T t^2\sigma_t^2\,dt$. Various types of diffusion can be determined from upper functions (see assertions (a)–(c) of Lemma 2). Indeed, the definition of the upper function in Lemma 2 involves, in addition to M_T, the quadratic characteristic $\widehat{M}_T = \int_0^T t^2\ln^2(t+1)\,\sigma_t^2\,dt$. Assertions (a) and (b) of Lemma 2 produce a subdiffusion, which agrees with the condition $M_T/T \to 0$, $T \to \infty$. However, under this approach, a subdiffusion (if $(\ln^2 T)M_T/T \to 0$), a normal diffusion (if $\widehat{M}_T \sim T$, $(\ln T)M_T/T \to 0$), or a superdiffusion (if $\widehat{M}_T/T \to \infty$, $T \to \infty$) may correspond to the upper function from assertion (c). Passing to the conditions of assertion (1) in Theorem 3, we see that $c_1 \leqslant t\mathbf{E}X_t^2 \leqslant c_2$ ($c_1, c_2 > 0$ are some constants) and $c_1 T \leqslant M_T \leqslant c_2 T$; that is, the upper function of assertion (c) is found to be related to a superdiffusion, since $h_T \sim \ln T\sqrt{T\ln\ln T}$, which is also in disagreement with the normal diffusion type, as detected from the mean square displacement. For assertion (3) of Theorem 3, that is, in the case $t\mathbf{E}X_t^2 \to \infty$, $t \to \infty$, both approaches lead to a superdiffusion. The above disagreements for $\kappa = -1$ are due to the presence of the terms $\widehat{I}_T = \int_0^T \widehat{m}_t\sigma_t\,dw_t$ and $m_t I_t = m_t\int_0^T t\sigma_t\,dw_t$ in representation (13) for the displacement process with more rapidly increasing quadratic variations.

Next, we proceed with the analysis of the case $-1 < \widetilde{\kappa} \leqslant 0$ of Assumption $\mathscr{D}$. The relation $\mathbf{E}X_t^2/\delta_t$, which appears under the hypothesis of Theorem 3, can be written in a more convenient form as follows:

$$\frac{\mathbf{E}X_t^2}{\delta_t} = \frac{1}{z_t}\int_0^t \exp\biggl\{2\int_0^s \delta_v\,dv\biggr\}\sigma_s^2\,ds; \tag{26}$$

here the function $z_t = \delta_t\exp\bigl\{2\int_0^t \delta_v\,dv\bigr\}$ satisfies the equation

$$\dot{z}_t = (2+\phi_t)\delta_t z_t, \qquad \phi_t = \frac{\dot{\delta}_t}{\delta_t^2}. \tag{27}$$

In the case $\widetilde{\kappa} = 0$, that is, when $\phi_t \to -\infty$, $t \to \infty$, it readily follows that

$$z_t \to 0, \quad \mathbf{E}X_t^2/\delta_t \to \infty \quad \text{as} \quad t \to \infty,$$

and hence by assertion (3) of Theorem 3, the process is a superdiffusion in the mean square sense. The upper functions h_T of the process are given by Theorem 2, and, moreover, we have $h_T \geqslant h_T^{(1)}$, where $h_T^{(1)} = Tg_T\sqrt{\mathbf{E}X_T^2}$. Note that $T\delta_T \to 0$, $T \to \infty$, since $\widetilde{\kappa} = 0$. More precisely, we have

$$T^2\mathbf{E}X_T^2 \geqslant cT^2\exp\biggl\{-2\int_0^t \delta_v\,dv\biggr\} \geqslant \widetilde{c}\,T^{2-\varepsilon},$$

where $\varepsilon > 0$ is a small number, and $c, \widetilde{c} > 0$ are some constants. Hence,

$$\frac{h_T^{(1)}}{\sqrt{T\ln T\ln T}} \geqslant \widehat{c}g_T\sqrt{\frac{T^{1-\varepsilon}}{\ln\ln T}} \to \infty, \qquad T \to \infty,$$

for some constant $\widehat{c} > 0$. So, we also have a superdiffusion according to the upper functions. A similar argument also holds for $-1/2 < \widetilde{\kappa} < 0$. As a result, $z_t \to 0$, $t \to \infty$, and we have $\delta_t \sim 1/t$ and $T\delta_T \to -\widetilde{\kappa}$, $T \to \infty$, for the stability rate. Moreover, $h_T \geqslant h_T^{(1)}$, where $h_T^{(1)} = Tg_T\sqrt{\mathbf{E}X_T^2}$ and $T^2\mathbf{E}X_T^2 \geqslant cT^{2(1+\widehat{\kappa})}$ with some constants $c > 0$ and $-1/2 < \widehat{\kappa} < \widetilde{\kappa} < 0$, which implies that

$$\frac{h_T^{(1)}}{\sqrt{T \ln T \ln T}} \geqslant g_T \sqrt{\frac{cT^{1+2\widehat{\kappa}}}{\ln\ln T}} \to \infty, \qquad T \to \infty;$$

in this case, a superdiffusion also takes place. In the next theorem, we combine the above results of the case $-1/2 < \widetilde{\kappa} \leqslant 0$ with the case $-1 < \widetilde{\kappa} < -1/2$.

THEOREM 5. *Let Assumption* $\mathscr{D}$ *hold with constant* $-1 < \widetilde{\kappa} \leqslant 0$, $\widetilde{\kappa} \neq 1/2$. *Then the types of diffusions, as detected from the dynamics of the mean square displacement and from the upper functions, coincide and can be specified from assertions* (1)–(3) *of Theorem* 3 (*the quantities* d_1^* *and* d_2^* *are given in Theorem* 3) *as follows*:

(1) *a normal diffusion for* $0 < d_1^* \leqslant d_2^* < \infty$,
(2) *a subdiffusion for* $d_2^* = 0$,
(3) *a superdiffusion for* $d_1^* = \infty$.

Proof. It remains to consider the case $-1 < \widetilde{\kappa} < -1/2$. Hence in (27) we have

$$2 + \phi_t \to 2 + \frac{1}{\widetilde{\kappa}} > 0, \quad t \to \infty, \quad \text{and} \quad z_t \to \infty, \quad t \to \infty,$$

which shows that the conditions of assertions (1)–(3) of Theorem 3 may be satisfied. Using representation (22) for $\mathbf{E}\widetilde{X}_t^2$, and taking into account that $\delta_t \sim 1/t$, we find that

$$\widetilde{c}\int_0^T t^2\sigma_t^2\,dt = \mathbf{E}\widetilde{X}_T^2 + 2\int_0^T \widetilde{\delta}_t \mathbf{E}\widetilde{X}_t^2\,dt, \quad \frac{\widetilde{\delta}_t}{\delta_t} \to 1 + \frac{1}{\widetilde{\kappa}} < 0, \quad t \to \infty, \tag{28}$$

where $\widetilde{c} > 0$ is a constant. It is clear that $d_2^* = 0$ (see assertion (2)); that is, $\mathbf{E}\widetilde{X}_T^2/T \to 0$, $T \to \infty$, and so it follows from (28) that $T^{-1}\int_0^T t^2\sigma_t^2\,dt \to 0$, $T \to \infty$. The upper function h_T, as given by Theorem 2, can be estimated as

$$h_T \leqslant h_T^{(1)} + h_T^{(2)}, \tag{29}$$

where

$$h_T^{(1)} \sim T\sqrt{\mathbf{E}X_T^2 \ln\ln T} + T\sqrt{\mathbf{E}X_T^2 \ln\ln \int_0^T \sigma_t^2\,dt},$$
$$h_T^{(2)} \sim \sqrt{\int_0^T t^2\sigma_t^2\,dt\, \ln\ln \int_0^T t^2\sigma_t^2\,dt}.$$

We have $\int_0^T t^2\sigma_t^2\,dt/T \to 0$, $T \to \infty$, and hence $h_T^{(1)} \sim T\sqrt{\mathbf{E}X_T^2 \ln\ln T}$ and

$$\limsup_{T\to\infty} \frac{h_T^{(1)}}{\sqrt{T\ln\ln T}} = \limsup_{T\to\infty} \frac{h_T^{(2)}}{\sqrt{T\ln\ln T}} = 0.$$

As a result, $h_T/\sqrt{T\ln\ln T} \to 0$, $T \to \infty$; that is, subdiffusion takes place.

We have $\int_0^T \exp\{2\int_0^t \delta_v\,dv\}\sigma_t^2\,dt \to \infty$ as $T \to \infty$ by the assumption $d_1^* = \infty$ of assertion (3), and so, for the upper function h_T from assertion (c) of Theorem 2, we have

$$\frac{h_T}{\sqrt{T\ln\ln T}} \geqslant \frac{h_T^{(1)}}{\sqrt{T\ln\ln T}} \geqslant T\frac{\sqrt{\mathbf{E}X_T^2\ln\ln T}}{\sqrt{T\ln\ln T}} \to \infty, \qquad T \to \infty,$$

which corresponds to the upper function of superdiffusion.

Finally, under the hypothesis of assertion (1) we have $c_1 \leqslant t\mathbf{E}X_t^2 \leqslant c_2$ ($c_1, c_2 > 0$ are some constants), and, moreover, $\int_0^T \exp\{2\int_0^t \delta_v\,dv\}\sigma_t^2\,dt \to \infty$, $T \to \infty$. Besides, using (28), we find that $T^{-1}\int_0^T t^2\sigma_t^2\,dt \leqslant \widetilde{c}$, where $\widetilde{c} > 0$ is a constant. Hence, from assertion (c) of Theorem 2 we have the estimates for the upper function

$$\widetilde{c}_1 h_T^{(1)} \leqslant h_T \leqslant \widetilde{c}_2 h_T^{(1)} + \widetilde{c}_2\sqrt{T\ln\ln T}$$

with constants $\widetilde{c}_1, \widetilde{c}_2 > 0$ and the function $h_T^{(1)} = T\sqrt{\mathbf{E}X_T^2\ln\ln T}$. Clearly, $0 < \liminf_{T\to\infty}(h_T^{(1)}/\sqrt{T\ln\ln T}) \leqslant \limsup_{T\to\infty}(h_T^{(1)}/\sqrt{T\ln\ln T}) < \infty$, and hence the diffusion is normal. Theorem 5 is proved.

The case $\widetilde{\kappa} = -1/2$ is considered separately below. Let $\delta_t = 1/(2t)$. Then the function z_t in (26) is constant (see (27)), and, from the point of the mean square displacement, just the normal diffusion (assertion (1) of Theorem 3) and the superdiffusion (assertion (3) of Theorem 3) are possible. The relation in assertion (1) of Theorem 3 also implies the condition $\int_0^\infty \exp\{2\int_0^t \delta_v\,dv\}\sigma_t^2\,dt = \int_0^\infty t\sigma_t^2\,dt < \infty$, and so the form of the upper function of the displacement process is specified in assertions (a) and (b) of Theorem 2. We also show that

$$\frac{1}{T}\int_0^T t^2\sigma_t^2\,dt = \int_0^T t\sigma_t^2\,dt - \frac{1}{T}\int_0^T\int_0^t s\sigma_s^2\,ds\,dt \to 0, \qquad T \to \infty.$$

Moreover, it is clear that $h_T/\sqrt{T\ln\ln T} \to 0$, $T \to \infty$; that is, a subdiffusion is detected from the upper functions, which is in disagreement with the normal-type diffusion based on the analysis of the mean square displacement. Under the hypotheses of assertion (3) of Theorem 3 on superdiffusion, the same diffusion type (that is, a superdiffusion) can also be detected from the analysis of upper functions: we have at our disposal assertion (c) of Theorem 2 with

$$h_T \geqslant ch_T^{(1)},$$

where $h_T^{(1)} \sim T\sqrt{\mathbf{E}X_T^2}\sqrt{\ln\ln T}$ and $h_T^{(1)}/\sqrt{T\ln\ln T} \to \infty$, $T \to \infty$.

In the next example we study the matching of diffusion types, as determined from the mean square displacement and upper functions in the case of a power-law family of coefficients (the stability rate and the diffusion coefficient).

Example 4. Let $\delta_t \sim t^k$, $\sigma_t^2 \sim t^m$, where k, m are real numbers, $k > -1$. Particular cases of such a diffusion are, for example, the physical processes from [19], [20] and the cognitive model from [21]. The function ϕ_t reads as $\phi_t = \dot{\delta}_t/\delta_t^2 \sim -kt^{-k-1}$, and $\phi_t \to 0$, $t \to \infty$, for $k > -1$; that is, Assumption $\mathscr{D}$ holds with $\kappa = 0$. The upper function of the process is obtained from Theorem 1. If $m < 2k - 1$, then $\int_0^\infty (\sigma_t^2/\delta_t^2)\,dt = \int_0^\infty t^{m-2k}\,dt < \infty$, and hence, according to assertion (a) of Theorem 1, the mean square displacement D_∞ is finite, and the adjusted displacement process $Y_T^{(\psi)}$ (see (9)) converges to $Y_\infty^{(\psi)} = \int_0^\infty t^{m-k}\, dw_t$ with probability 1 as $T \to \infty$. If $m \geqslant 2k-1$, then the upper function is of the form (11). Taking into account the relations $\mathbf{E}X_t^2 \sim t^{m-k}$ and $\mathbf{E}X_t^2/\delta_t \sim t^{m-2k}$ as $t \to \infty$, we get the following types of diffusions from Theorem 4: a subdiffusion for $m < 2k$, a normal diffusion for $m = 2k$, and a superdiffusion for $m > 2k$. Indeed, for the mean square displacement, we have $D_T \sim \int_0^T t^{m-2k}\, dt$, and hence $D_T \sim T^{m-2k+1}$ if $m-2k \neq -1$, and $D_T \sim \ln T$ if $m-2k=-1$. Then, for $m < 2k$, we also have a subdiffusion; a normal diffusion appears in the case $m = 2k$, and a superdiffusion appears in the case $m > 2k$. Using (11) to derive the upper functions, we find that $\mathbf{E}X_t^2/\delta_T^2 \sim T^{m-3k}$, $\ln \int_0^T \delta_t\, dt \sim T$, the quadratic characteristic assuming the form $\int_0^T (\sigma_t^2/\delta_t^2)\, dt \sim T^{m-2k+1}$ if $m-2k \neq -1$, and $\int_0^T (\sigma_t^2/\delta_t^2)\, dt \sim \ln T$ if $m-2k=-1$. Hence

$$h_T \sim h_T^{(1)} + h_T^{(2)},$$

where

$$h_T^{(1)} = \sqrt{T^{m-3k}\ln T}, \qquad h_T^{(2)} = \begin{cases} \sqrt{T^{m-2k+1}\ln\ln T}, & m-2k \neq -1, \\ \sqrt{\ln T \ln\ln\ln T}, & m-2k=-1. \end{cases}$$

As $h_T^{(1)}/h_T^{(2)} \to 0$, $T \to \infty$, we have $h_T \sim h_T^{(2)}$ and

$$h_T \sim D_T \ln\ln D_T, \quad \text{where} \quad D_T \sim \begin{cases} T^{m-2k+1}, & m-2k \neq -1, \\ \ln T, & m-2k=-1. \end{cases}$$

So, the results of classification of diffusions types from upper functions agree with those obtained from the mean square displacement (see also Theorem 4).

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