# On Infinite Time Linear–Quadratic Gaussian Control of Inhomogeneous Systems*

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***Abstract*— In this paper we consider infinite-time linear quadratic Gaussian control problems for non-autonomous systems with possibly unbounded coefficients of their inhomogeneous components. It is shown that under standard stabilizability and detectability conditions, along with assumptions on inhomogeneous term growth, there exists an optimal control law. This control law of well-known feedback form appears to be optimal with respect to proposed extended criteria. The applied criteria expand the notions of long-run average and pathwise long-run average, commonly used for ergodic control. We also investigate the discounted tracking problems, where the cost incorporates a discount function penalizing deviations from reference trajectories.**

## I. INTRODUCTION

The linear system theory has been recognized one of the most intensively studied fields with a wide range of applications. Stochastic control problems posed over an infinite-time horizon are of particular interest involving, along with investigation of basic system properties such as stabilizability, controllability etc., some specific issues. As the commonly used long-run average cost does not always conform, the first important task which seems to arise is the choice of optimization criterion. Being in a stochastic setting means that we may also like to consider pathwise control problems, e.g., pathwise ergodic control, developed mainly for time-invariant systems [1]–[3]. However, in economic applications time-invariance could not be perfectly adopted, at least if we include non-exponential discounting in the model. Again, long-run macroeconomic policy evaluation, assuming integral quadratic performance index, requires the use of target states [4], [5], as well as possibly unbounded reference trajectories that vary in time [6], [7] (for financial and insurance applications see [8], [9]). This paper deals with infinite-time linear quadratic Gaussian control for non-autonomous systems having inhomogeneous part both in the state equation and cost. In contrast to the standard LQG setting, we permit unbounded growth of inhomogeneous components as time goes to infinity. For such a class of control systems the long-run average cost becomes irrelevant, not providing finite values even on stable feedback laws. Therefore, one of the main contributions of this work consists in development of an extended criterion within the study of average optimality. The average optimality concept, e.g., in Markov Decision Processes, see [10], emphasizes that the problem of seeking an optimal control relies on some criterion associated with expected costs. The essential feature of infinite-time LQG control is that the form of an average optimal strategy should be derived as a limit of solutions obtained on bounded time intervals. This implies the consideration of the stable linear feedback law, which we prove to be optimal. In stochastic environment optimality on the average may not represent the best potential long-term performance characteristic of a control. Instead, one might wish to consider a criterion based on sample paths of stochastic processes. It is calculated pathwise, and, in general, defines a random variable (e.g., pathwise long-run average or pathwise ergodic costs, see [2], [3]). Here we propose an extended pathwise criterion for inhomogeneous systems. Another contribution of our research is that the optimal feedback is also shown to possess the pathwise optimality property under conditions on criterion denominator. The case of discounted tracking problem is also considered, when the cost contains asymptotically singular matrices and possibly unbounded reference trajectories. The work is organized as follows. In Section 2 we introduce the linear inhomogeneous control system together with basic assumptions which guarantee the existence of optimal feedback law. Section 3 presents our main results regarding the optimality of the feedback with respect to extended criteria. Section 4 is devoted to the discounted control problems. The last section concludes.

## II. PROBLEM FORMULATION

Let $X_t,\ t \geq 0$, be an $\mathsf{R}^n$–valued process defined on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ by

$$dX_t = A_t X_t dt + B_t U_t dt + m_t dt + G_t dw_t\,, \quad X_0 = x\,, \tag{1}$$

where $x$ is a non-random vector; $A_t,\ B_t,\ G_t,\ t \geq 0$, are bounded (non-random) time-varying matrices of appropriate dimensions; $w_t, t \geq 0$ is a $d$-dimensional standard Wiener process; $U_t,\ t \geq 0$, is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\ s \leq t\}$–adapted $k$-dimensional process such that there exists a solution to (1). Let us denote by $\mathcal{U}$ the set of admissible controls. The quadratic cost over the planning horizon $[0, T]$ is given by

$$J_T(U) = \int_0^T (X_t' Q_t X_t + U_t' R_t U_t + 2q_t' X_t + 2r_t' U_t)\, dt, \tag{2}$$

where $U \in \mathcal{U}$ is an admissible control on $[0, T]$, $Q_t \geq 0$, $R_t \geq \rho I,\ t \geq 0$, are bounded symmetric time-varying matrices. Here $'$ denotes the matrix transpose, $\rho > 0$ is a constant, we write $A \geq B$ if the difference $A - B$ is positive

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semidefinite, $I$ is an identity matrix. We allow the vector functions $m_t$, $r_t$, $q_t$ describing inhomogeneous terms to be unbounded as $t \to \infty$. More specifically, we adopt the following assumption:

**Assumption 1.** Let $L_t = \|m_t\|^2 + \|q_t\|^2 + \|r_t\|^2$ with $\|\cdot\|$ stated for the Euclidean matrix norm. Then

$$\lim_{t\to\infty} \frac{L_t}{\int_0^t L_s\, ds} = 0\,.$$

By Assumption 1 possible extreme growth of inhomogeneous part is avoided. We also need to assume that the standard requirements concerning system properties are satisfied.

**Assumption 2.** $(A_t, B_t)$ is stabilizable and $(A_t, \sqrt{Q_t})$ is detectable (for the definitions of these properties refer to, e.g., [11]).

Assumption 2 implies, see, e.g., [12], that there exists a bounded symmetric solution $\Pi_t \geq 0$, $t \geq 0$, to the matrix Riccati differential equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t'\Pi_t - \Pi_t B_t R_t^{-1} B_t'\Pi_t + Q_t = 0\,, \qquad (3)$$

such that the matrix $\mathcal{A}_t = A_t - B_t R_t^{-1} B_t'\Pi_t$ is exponentially stable, i.e. the transition matrix $\Phi(t,s)$ corresponding to $\mathcal{A}_t$ admits the upper bound

$$\|\Phi(t,s)\| \leq \kappa_1 e^{-\kappa_2(t-s)}, \qquad s \leq t\,,$$

with some constants $\kappa_1$, $\kappa_2 > 0$.

Then we can define the so-called *stable feedback control law* $U^*$ in the form

$$U_t^* = -R_t^{-1}[r_t + B_t'(\Pi_t X_t^* + p_t)]\,, \qquad (4)$$

where $p_t$ is the solution to

$$\dot{p}_t + \mathcal{A}_t' p_t + \Pi_t(m_t - B_t R_t^{-1} r_t) + q_t = 0\,, \qquad (5)$$

and the process $X_t^*$, $t \geq 0$, is governed by

$$dX_t^* = \mathcal{A}_t X_t^* dt + \mu_t dt + G_t dw_t\,, \quad X_0^* = x\,. \qquad (6)$$

with $\mu_t = m_t - B_t R_t^{-1}(r_t + B_t' p_t)$.

In the autonomous case of (1)–(2), the stable feedback $U^*$ happens to be optimal with respect to the long-run average cost criterion [13]. Namely, $U^*$ solves

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{T} \to \inf_{U\in\mathcal{U}}\,.$$

The above criterion does not seem to be suitable for the inhomogeneous system, e.g., being unbounded when $L_t \to \infty$, $t \to \infty$, or taking zero value if $\|G_t\|^2 + L_t \to 0$, $t \to \infty$. Thus the long-run average, as well as its respective pathwise (ergodic) modification, needs to be extended in order to cover the inhomogeneous case. The control problem with pathwise long-run average

$$\limsup_{T\to\infty} \frac{J_T(U)}{T} \to \inf_{U\in\mathcal{U}} \quad \text{almost surely (a.s.)} \qquad (7)$$

also has $U^*$ as a solution if the linear control system is time-invariant [14]. The notion *pathwise* comes from the fact that

$$\limsup_{T\to\infty}(J_T(U^*)/T) \leq \limsup_{T\to\infty}(J_T(U)/T)$$

holds $P$-almost surely. The probability measure $P$ was previously introduced to define the Wiener process $w_t$ on $\{\Omega, \mathcal{F}, \mathbf{P}\}$, see (1). Here and in what follows we adopt the notation *a.s.* (almost surely) to indicate a particular relation valid with probability 1 when $P$ is used (e.g. convergence of stochastic processes).

It is worthwhile to mention previous studies [15], [16] on pathwise LQG control with inhomogeneous terms. The primary focus there was on the cost difference growth as $T \to \infty$. The problem of tracking periodic function under considerable restrictions on the set of admissible controls (in particular, those providing bounded variance of the state process) admitted *a.s. overtaking optimal* $U^*$, i.e. $J_T(U) - J_T(U^*) \to \infty$, a.s., $T \to \infty$, see [15]. In the case of (1)–(2) with bounded vector functions, it was proven [16] that $\limsup_{T\to\infty} g_T(J_T(U^*) - J_T(U)) \leq 0$ a.s. for $g_T = o(1/\ln T)$, meaning $g_T$ is infinitesimal compared with $1/\ln T$ as $T \to \infty$. In the next section we will investigate not only the cost difference asymptotic behavior, but also the cost $J_T(U^*)$ itself in order to establish appropriate average and pathwise optimality criteria.

## III. ON AVERAGE AND PATHWISE OPTIMALITY OF THE STABLE FEEDBACK

In this section we consider infinite horizon control problems for (1)–(2).

The criterion used here has more general form than the well-known long-run average, so we call it the *extended long-run average* and have to solve

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T (\|G_t\|^2 + L_t)\, dt} \to \inf_{U\in\mathcal{U}}\,. \qquad (8)$$

As it was pointed out in Section 2 regarding linear autonomous systems, in fact we are to study an *average optimality over an infinite time-horizon* of the feedback $U^*$. Generally speaking, one may define the average optimality of control as being established with respect to an index based on expected values of the cost (e.g. the long-run average cost). The problem for homogeneous case, $L_t \equiv 0$, was treated in [17] with the impact of $G_t$ included into the criterion.

Next we provide two useful lemmas concerning bounds on $p_t$ and $X_t^*$ in terms of $L_t$.

**Lemma 1.** Let Assumptions 1 and 2 be satisfied. Then

a) $\int_0^t \|p_s\|^2\, ds \leq c_L \int_0^t L_s\, ds$ for some constant $c_L > 0$;

b)

$$\lim_{t\to\infty} \frac{\|p_t\|^2}{\int_0^t L_s\, ds} = 0\,.$$

**Proof.** Obviously,

$$p_t = \int_t^\infty \Phi'(s,t)[q_s + \Pi_s(m_s - B_s R_s^{-1} r_s)]\, ds$$

is a solution to (5). Using the Cauchy-Schwarz inequality and the exponential stability property of $\mathcal{A}_t$ , we obtain

$$P_t = \int_t^\infty e^{-\kappa_2(s-t)} L_s\, ds$$

and $\|p_t\|^2 \le c P_t$ , for some constant $c > 0$. The integration of the latter by parts gives a).

Applying Assumption 1, we may write for $\epsilon < \kappa_2$ and $t > t_0(\epsilon)$

$$L_t < \epsilon \int_0^t L_s\, ds\,, \quad P_t < \frac{\epsilon}{\kappa_2 - \epsilon} \int_0^t L_s\, ds\,,$$

which implies that $P_t / \int_0^t L_s\, ds \to 0,\ t \to \infty$ and, together with the estimate previously obtained for $\|p_t\|^2$, gives b).

**Lemma 2.** Let Assumptions 1 and 2 be satisfied. Then

$$\lim_{t\to\infty} \frac{\|X_t^*\|^2}{\int_0^t (\|G_s\|^2 + L_s)\, ds} = 0 \quad \text{almost surely}\,.$$

**Proof.** We express the solution of (6) in the form $X_t^* = Z_t + \beta_t$, where $Z_t$ solves the homogeneous stochastic differential equation (6), i.e. for $m_t \equiv q_t \equiv r_t \equiv 0$, and

$$\beta_t = \int_0^t \Phi(t,s)[m_t - B_t R_t^{-1}(r_t + B_t' p_t)]\,.$$

Then from Assumptions 1–2, b) of Lemma 1 and for some constant $c > 0$ it follows that

$$\|\beta_t\|^2 \le c \int_0^t e^{-\kappa_2(s-t)} L_s\, ds\,. \tag{9}$$

Based on (9), by the similar argument as in the proof of Lemma 1, we have $\|\beta_t\|^2 / \int_0^t L_s\, ds \to 0,\ t \to \infty$. As concerns $Z_t$, it is known (see [18]) that

$$\frac{\|Z_t\|^2}{\int_0^t \|G_s\|^2\, ds} \to 0\,, \quad \text{a.s., } t \to \infty\,,$$

and this concludes the proof of the lemma.

Detectability of $(A_t, \sqrt{Q_t})$ required in Assumption 2 implies state-dependent inequality for deterministic homogeneous linear regulator with zero initial condition. It will be needed later to prove the optimality of the feedback $U^*$.

**Lemma 3.** Consider $(x_t, u_t)$, satisfying the deterministic linear system

$$dx_t = A_t x_t dt + B_t u_t dt\,, \quad x_0 = 0\,. \tag{10}$$

If $(A_t, \sqrt{Q_t})$ is detectable, then there exists a constant $c_0 > 0$, such that for any $(x_t, u_t)$ obeying (10), and $T > 0$, we have

$$\|x_T\|^2 + \int_0^T \|x_t\|^2\, dt \le c_0 \int_0^T (x_t' Q_t x_t + u_t' R_t u_t)\, dt\,.$$

The validity of the above statement may be established by straightforward adaptation of the argument from the proof of Lemma 3.6 in [12] using the detectability property of $(A_t, \sqrt{Q_t})$, and hence we omit the proof here.

Now we are ready to state the main result on the average optimality of the feedback $U^*$.

**Theorem 1.** Let Assumptions 1 and 2 be satisfied. Then the control $U^*$ given by (4)–(6) is a solution to (8).

**Proof.** Choose $U \in \mathcal{U}$ and define $u_t = U_t - U_t^*$, $x_t = X_t - X_t^*$. Note that $(x_t, u_t)$ satisfies (10). It may be shown, see also [16], that $J_T(U^*) - J_T(U) =$

$$-2\int_0^T [(X_t^*)' Q_t x_t + (U_t^*)' R_t u_t + q_t' x_t + r_t' u_t]\, dt$$

$$-\int_0^T (x_t' Q_t x_t + u_t' R_t u_t)\, dt = 2x_T'(\Pi_T X_T^* + p_T)$$

$$-\int_0^T (x_t' Q_t x_t + u_t' R_t u_t)\, dt - 2\int_0^T x_t' \Pi_t G_t\, dw_t$$

$$\le c_1(\|X_T^*\|^2 + \|p_T\|^2) - c_2 \int_0^T \|x_t\|^2\, dt - 2\int_0^T x_t' \Pi_t G_t\, dw_t. \tag{11}$$

The last inequality (11), valid for some constants $c_1, c_2 > 0$, was obtained by the use of Lemma 3 and the elementary inequality $2ab \le a^2/c + cb^2$ for appropriate $a, b$, and $c > 0$. Thus we have

$$E J_T(U^*) \le E J_T(U) + c_1(E\|X_T^*\|^2 + \|p_T\|^2) \tag{12}$$

After dividing both sides of (12) by $\int_0^T (\|G_t\|^2 + L_t)\, dt$, apply Lemmas 1–2 and get the required optimality of $U^*$.

Recall (8). We conclude the proof showing that the extended long-run average criterion is finite on $U^*$. From (3)–(6) we obtain the expression for the cost

$$J_T(U^*) = x'\Pi_0 x + 2p_0' x$$
$$-(X_T^*)'\Pi_T X_T^* - 2p_T' X_T^* + N_T + 2M_T\,, \tag{13}$$

where $N_T = \int_0^T tr(G_t' \Pi_t G_t)\, dt$

$$+\int_0^T [2p_t'(\mu_t + B_t R_t^{-1} B_t' p_t) - r_t' R_t^{-1} r_t]\, dt\,,$$

$tr(\cdot)$ denotes the matrix trace and the term $M_T$ is of the form

$$M_T = \int_0^T (X_t^*)'\Pi_t G_t dw_t + \int_0^T p_t' G_t dw_t \,,$$

From the above expressions and the use of Lemmas 1–2 we get the finite value of $\limsup_{T\to\infty} EJ_T(U^*)/\int_0^T (\|G_t\|^2 + L_t)\,dt$. The proof is complete.

Below we study a stochastic control problem with *extended pathwise long-run average criterion*

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T (\|G_t\|^2 + L_t)\,dt} \to \inf_{U\in\mathcal{U}} \quad \text{a.s.}\,, \qquad (14)$$

which is a straightforward generalization of (7). The notion of *pathwise* control means that we apply optimality criteria constructed on sample paths themselves rather than on their average values.

Afterwards the following assumption appears to be necessary in the study of pathwise optimality:

**Assumption 3.**

$$\lim_{T\to\infty} \frac{1}{\int_0^T (\|G_t\|^2 + L_t)\,dt} = 0\,.$$

Note we could not expect Assumption 3 to be a corollary of Assumption 1. For instance, it may not hold for $L_t$ and $G_t$ that tend to zero as $t\to\infty$.

**Theorem 2.** Let Assumptions 1–3 be satisfied. Then the control $U^*$ given by (4)–(6) is a solution to (14). Moreover,

$$\limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T (\|G_t\|^2 + L_t)\,dt} = \limsup_{T\to\infty} \frac{EJ_T(U^*)}{\int_0^T (\|G_t\|^2 + L_t)\,dt}\,. \qquad (15)$$

**Remark 1.** Note that (15) can be viewed as analogous to the well-known ergodicity property in pathwise stochastic control of autonomous systems [1].

**Proof.** Recall the upper bound (11) previously established for the difference $J_T(U^*) - J_T(U)$ and notice that

$$\limsup_{T\to\infty} g_T(c_2 \int_0^T \|x_t\|^2\,dt - 2\int_0^T x_t'\Pi_t G_t\,dw_t) \le 0$$

for any function $g_T \to 0$, $T\to\infty$, please refer to Lemma A.1 in [16]. Since Assumption 3 holds, we may set $g_T = 1/\int_0^T (\|G_t\|^2 + L_t)\,dt$. Then the pathwise optimality of $U^*$ follows from Lemmas 1 and 2.

We use (13) to prove (15). It is sufficient to study the asymptotic behavior of the Ito integral $M_T$. Seeing that its quadratic variation

$$EM_T^2 \le c\int_0^T (\|G_t\|^2 + L_t)\,dt\,, \quad \text{with some } c>0\,,$$

by Lemma 1 from [18] we get $M_T/\int_0^T (\|G_t\|^2 + L_t)\,dt \to 0$, a.s., $T\to\infty$. Thus the proof is finished.

## IV. DISCOUNTED TRACKING PROBLEMS

Consider a linear time-invariant controlled stochastic process $Y_t$, $t\ge 0$ :

$$dY_t = AY_t dt + BV_t dt + G dw_t\,, \qquad Y_0 = y\,, \qquad (16)$$

where $y$ is non-random, $V_t$, $t\ge 0$ is an admissible control defined similarly to that of Section 2; $\mathcal{V}$ is the set of admissible controls.

Let vector functions $y_t$, $v_t$ represent the reference trajectories of state and control, respectively. Assume that any deviation of $Y_t$, $V_t$ from $y_t$, $v_t$ results in a loss. The loss is evaluated by the agents according to the concept of time preference which relates to the timing of an outcome and can be expressed by means of discount function $f_t>0$. The total loss over the planning horizon $[0,T]$ is measured by a quadratic cost functional:

$$J_T^{(d)}(V) = \int_0^T f_t[(Y_t-y_t)'Q(Y_t-y_t)+(V_t-v_t)'R(V_t-v_t)]\,dt\,, \qquad (17)$$

where $A,B,G,Q\ge 0,R>0$, are known constant matrices; $V\in\mathcal{V}$; non-increasing $f_t$ is a discount function, $\lim_{t\to\infty} f_t = 0$, $f_0 = 1$, with bounded discount rate $\phi_t = -\dot{f}_t/f_t$.

*Examples of discount functions.* Conventional exponential discounting $f_t = e^{-\gamma t}$, $\gamma>0$; hyperbolic discounting [19] $f_t = 1/(1+\theta t)^{\theta_1/\theta}$, for $\theta,\theta_1>0$; double exponential [20] $f_t = \lambda e^{-\gamma_1 t} + (1-\lambda)e^{-\gamma_2 t}$; $\gamma_1,\gamma_2>0$, $0\le\lambda\le 1$.

In the absence of reference paths known criterion for discounted LQG control is based on *long-term expected loss per unit of cumulated discount* [21]:

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(V)}{\int_0^T f_t\,dt} \to \inf_{V\in\mathcal{V}}\,.$$

This basic criterion was firstly introduced in [22] for controlled diffusion processes in bounded domains with no relation to linear-quadratic control. Taking into account the presence of reference trajectories, we denote by $S_t = \|y_t\|^2 + \|v_t\|^2 + \|G\|^2$ and solve the control problem with *extended criterion of long-run expected loss* per unit of cumulated discount:

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(V)}{\int_0^T f_t S_t\,dt} \to \inf_{V\in\mathcal{V}}\,, \qquad (18)$$

subsequently introducing the pathwise control problem

$$\limsup_{T\to\infty} \frac{J_T^{(d)}(V)}{\int_0^T f_t S_t\,dt} \to \inf_{V\in\mathcal{V}} \quad \text{a.s.} \qquad (19)$$

The denominator in the extended criteria can be justified by the following argument. Because of the cost matrices, (16)–(17) is non-standard compared with (1)–(2) of Section 2. Let us perform change of variables

$$X_t = \sqrt{f_t} Y_t\,, \quad U_t = \sqrt{f_t} V_t\,,$$

immediately arriving to (1)–(2), with $A_t = A - (1/2)\phi_t I$, $B_t = B$, $G_t = \sqrt{f_t} G$, $m_t = 0$, $Q_t = Q$, $R_t = R$, $q_t = -\sqrt{f_t} Q y_t$, $r_t = -\sqrt{f_t} R v_t$ . For the costs we have

$$J_T^{(d)}(V) = J_T(U) + \int_0^T f_t(y_t' Q y_t + v_t' R v_t)\, dt\,.$$

Therefore the problem (18) can be eventually transformed into (8) for the new variables $X_t$, $U_t$. Let us rewrite Assumptions 1,3. It should be noticed that $L_t$ is then taken as $L_t = f_t(\|y_t\|^2 + \|v_t\|^2)$.

**Assumption 4.**

$$\lim_{t\to\infty} \frac{f_t(\|y_t\|^2 + \|v_t\|^2)}{\int_0^t f_s(\|y_s\|^2 + \|v_s\|^2)\, ds} = 0\,,$$

**Assumption 5.**

$$\lim_{t\to\infty} \frac{1}{\int_0^t f_\tau S_\tau\, d\tau} = 0\,.$$

**Remark 2.** Since $\phi_t \geq 0$, then the stabilizability of $(A, B)$ and detectability of $(A, \sqrt{Q})$ imply the respective properties for $(A - (1/2)\phi_t I, B)$ and $(A - (1/2)\phi_t I, \sqrt{Q})$.

Now we use (4)–(6), and by the inverse change of variables define

$$V_t^* = -R^{-1} B' (\Pi_t Y_t^* + \hat{p}_t) + v_t\,, \tag{20}$$

$$\hat{p}_t = \int_t^\infty \frac{f_s}{f_t} \Psi'(s,t)(\Pi_s B v_s - Q y_s)\, ds\,, \tag{21}$$

where $\Psi(t,s)$ corresponds to $\mathcal{A}_t = A - BR^{-1}B'\Pi_t$, the process $Y_t^*$, $t \geq 0$, is governed by

$$dY_t^* = \mathcal{A}_t Y_t^* dt + B(v_t - R^{-1} B' \hat{p}_t) dt + G dw_t\,, \quad Y_0^* = y\,. \tag{22}$$

Next we apply Theorems 1 and 2 to establish results on average and pathwise optimality of the given feedback $V^*$.

**Theorem 3.** Let $(A, B)$ be stabilizable, $(A, \sqrt{Q})$ detectable, and Assumption 4 holds. Then the control $V^*$ determined by (20)–(22) is a solution to (18). If, in addition, Assumption 5 is satisfied, then $V^*$ also solves the pathwise average control problem (19).

## V. CONCLUSIONS

We studied LQG optimal control problems over an infinite-time horizon. The stochastic control system possesses an exogenous component in the state equation and linear terms in the associated cost, so called inhomogeneous. Extending the notions of long-run average, two optimality criteria have been proposed, both having as denominator the integral quadratic index of system deviation from homogeneous deterministic case on the planning horizon [0,T]. Thus, average control and pathwise control problems were formulated. The suggested feedback form of the control law, known from deterministic LQ regulation, see, e.g., [23],[24], resulted to be optimal under quite standard assumptions, such as stabilizability and detectability, together with growth conditions on inhomogeneous part. We also established that the discounted tracking problems can be handled within the inhomogeneous LQG framework by a linear variable transformation. One direction for future research would be to consider control systems with stochastic inhomogeneous part. This is expected to require more complex treatment, e.g., giving rise to linear backward stochastic differential equations (BSDE) on infinite horizon.

## REFERENCES


[1] A. Arapostathis, V. S. Borkar, and M. K. Ghosh, Ergodic control of diffusion processes. Cambridge: Cambridge University Press, 2012, ch. 2.

[2] P. Dai Pra, G. B. Di Masi, and B. Trivellato, Almost sure optimality and optimality in probability for stochastic control problems over an infinite horizon, Ann. Oper. Res., vol. 88, pp. 161–171, 1999.

[3] P. Dai Pra, G. B. Di Masi, and B. Trivellato, Pathwise optimality in stochastic control, SIAM J. Control Optim., vol. 39, no. 5, pp. 1540–1557, 2000.

[4] J. K. Sengupta, Optimal Stabilization Policy with a Quadratic Criterion Function, Rev. Econ Stud., vol. 37, pp. 127–45, Jan. 1970.

[5] G. C. Chow, Control Methods for Macroeconomic Policy Analysis, The Am. Econ. Rev., vol. 66, pp. 340–345, May 1976.

[6] R. S. Pindyck, Optimal Policies for Economic Stabilization, Econometrica, vol. 41, pp. 529–560, May 1973.

[7] J. C. Engwerda, The solution of the infinite horizon tracking problem for discrete time systems possessing an exogenous component, J. Econ. Dyn. Control., vol. 14, no. 3–4, pp. 741–762, 1990.

[8] P. Mazzoleni, An Actuarial and Financial Analysis for Ecu Insurance Contracts, in Modelling for Financial Decisions. Berlin: Springer, 1991, pp. 139-155.

[9] M. Steffensen M., Differential systems in finance and life insurance, in Stochastic Economic Dynamics, B. S. Jensen, and T. Palokangas, Eds. Copenhagen: Copenhagen Business School Press, 2007, pp. 317–360.

[10] M. L. Puterman, Markov Decision Processes: Discrete Stochastic Dynamic Programming, New York: Wiley, 2005, pp. 334.

[11] A. Ichikawa, and H. Katayama, Linear Time Varying Systems and Sampled-data Systems. London: Springer, 2001, pp. 14.

[12] M. Mueller, and Cantoni M., Normalized coprime representations for time-varying linear systems, in Proc. 49th IEEE Conf. on Decision and Control, New York, 2010, pp. 7718–7723.

[13] H. Kwakernaak, and R. Sivan, Linear optimal control systems. New York: Wiley-interscience, 1972, ch. 3.

[14] E. Presman, V. Rotar, and M. Taksar, Optimality in probability and almost surely. The general scheme and a linear regulator problem, Stoch. Rep., vol. 43, no. 3/4, pp. 127–137, 1993.

[15] A. Leizarowitz, On almost sure optimization for stochastic control systems, Stochastics, vol. 23, no. 2., pp. 85–107, 1988.

[16] T. A. Belkina, Yu. M. Kabanov, and E. L. Presman, On a Stochastic Optimality of the Feedback Control in the LQG-Problem, Theor. Probab. Appl., vol. 48, no. 1, pp. 592–603, 2003.

[17] T. A. Belkina, and E. S. Palamarchuk, On Stochastic Optimality for a Linear Controller with Attenuating Disturbances, Autom. Rem. Control, no. 4, pp. 628–641, 2013.

[18] E. S. Palamarchuk, Asymptotic Behavior of the Solution to a Linear Stochastic Differential Equation and Almost Sure Optimality for a Controlled Stochastic Process, Comput. Math. Math. Phys., vol. 54, no. 1, pp. 83–96, 2014.

[19] G. Loewenstein, and D. Prelec, Anomalies in intertemporal choice: Evidence and an interpretation, Q. J. Econ., vol. 107, pp. 573–597, May 1992.

[20] S. M. McClure, K. M. Ericson, D. I. Laibson, G. Loewenstein, and J. D. Cohen, Time Discounting for Primary Rewards, J. Neurosci, vol. 27, no. 21, pp. 5796–5804, 2007.

[21] E. S. Palamarchuk, Stabilization of Linear Stochastic Systems with a Discount: Modeling and Estimation of the Long-Term Effects from the Application of Optimal Control Strategies, Math. Mod. and Computer Simulat., vol. 7, no. 4, pp. 381-388, 2015.

[22] A. Leizarowitz, Controlled Diffusion Processes on Infinite Horizon with the Overtaking Criterion, App. Math. Opt., vol. 17, no. 1, pp. 61–78, 1988.

[23] A. Bensoussan, Stochastic control of partially observable systems. Cambridge: Cambridge University Press, 2004, pp. 20–22.

[24] P. Dorato, C. T. Abdallah, and V. Cerone, Linear-quadratic control: an introduction. Malabar, FL: Krieger Publishing, 2000, ch. 3.