ISSN 0005-1179, Automation and Remote Control, 2020, Vol. 81, No. 1, pp. 53–63. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Avtomatika i Telemekhanika, 2020, No. 1, pp. 67–80.

STOCHASTIC SYSTEMS

Optimal Controller for a Nonautonomous Linear Stochastic System with a Two-Sided Cost Functional

E. S. Palamarchuk
Central Economics and Mathematics Institute, Moscow, Russia
e-mail: e.palamarchuck@gmail.com

Received May 31, 2019
Revised June 6, 2019
Accepted July 18, 2019

Abstract—The stochastic linear control problem over an infinite-time horizon with a two-sided cost functional and a time-varying diffusion matrix is considered. In the two-sided quadratic cost functional, the limits of integration have opposite sign and depend on the length of planning horizon. It is shown that under conditions on the diffusion matrix growth, the well-known linear feedback law is optimal in terms of the extended long-run average cost and its pathwise analog. In addition, the probabilistic behavior of the system's optimal path is studied.

Keywords: stochastic linear controller, two-sided cost functional, time-varying diffusion matrix

DOI: 10.1134/S0005117920010051

1. INTRODUCTION

Stochastic linear controllers belong to the class of control systems that are of theoretical and practical importance; see [1, Chapter 3]. Their dynamics are often considered on the positive semi-axis of the time parameter $t \in [t_0, T]$ and a planning horizon $[t_0, T] \subseteq [0, +\infty)$. At the same time, in a theoretical-operator perspective (i.e., in the case of infinite-dimensional state spaces; e.g., see [2]), the evolution of such systems can be analyzed on the entire real axis, $t \in (-\infty, +\infty)$. Then control problems are posed on the intervals $[t_0 - T, t_0 + T]$, where $T \geqslant 0$, with further letting $T \to +\infty$; see [3, 4]. Moreover, as was emphasized in [5], there exist applications (signal processing, statistical estimation, data transmission, and others) in which the models have the independent variable $t \in (-\infty, +\infty)$. Let us describe the control system studied in this paper. Consider a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ and let an n-dimensional stochastic process X_t, $t \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, be defined on this space according to the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dw_t, \tag{1}$$

where A_t and B_t are bounded matrices with time-varying entries; the disturbances are modeled by the so-called two-sided Wiener process w_t, $t \in \mathbb{R}$, defined in an usual way, i.e., $w_t = w_t^{(1)}$ for $t \geqslant 0$ and $w_t = w_{-t}^{(2)}$ for $t < 0$, where $w_t^{(1)}$ and $w_t^{(2)}$, $t \geqslant 0$, are two independent d-dimensional standard Wiener processes [6, p. 7]; the set of admissible controls $\mathcal{U}$ consists of the k-dimensional square integrable stochastic processes U_t, $t \in \mathbb{R}$, adapted to a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$, $\mathcal{F}_t = \sigma\{w_s, s \leqslant t\}$ ($\sigma(\cdot)$ denotes a σ-algebra) such that there exists a solution to Eq. (1), i.e., a process X_t, $t \in \mathbb{R}$, for which [3] the equality $X_t = X_s + \int_s^t A_\tau X_\tau \, d\tau + \int_s^t B_\tau U_\tau \, d\tau + \int_s^t G_\tau \, dw_\tau$ holds for all $s \leqslant t$ almost surely (a.s.); G_t is the diffusion matrix whose elements satisfy the assumptions below (for the time being, note that the disturbances parameters can be either bounded, e.g., constant $G_t \equiv G$ or damping $\|G_t\| \to 0$, or increasing $\|G_t\| \to \infty$, $t \to \pm\infty$, where $\|\cdot\|$ denotes the Euclidean matrix norm).

For $T > 0$, as a two-sided cost functional on $[-T, T]$ we defined the random variable

$$J_{2T}(U) = \int_{-T}^{T} (X_t' Q_t X_t + U_t' R_t U_t)\, dt, \tag{2}$$

where $U \in \mathcal{U}$ is an admissible control; $Q_t \geqslant qI$ and $R_t \geqslant \rho I$, $t \in \mathbb{R}$, are bounded symmetric matrices, $q, \rho > 0$ are some constants. (As usual, $'$ indicates the transpose; for matrices A and B, the relation $A \geqslant B$ means that their difference $A - B$ is a nonnegative definite matrix; I denotes an identity matrix.)

Previously, stochastic linear control problems on infinite horizon ($T \to +\infty$) with the cost functional (2) were considered in [7] subject to data transmission in networks and also in [8; 9, part 13.2.10] for to engineering applications. The optimality criterion was the long-run average cost, i.e., $\limsup\limits_{T\to+\infty}\{\mathrm{E}J_{2T}/(2T)\} \to \inf\limits_{U\in\mathcal{U}}$. Obviously, such an approach does not take into account the time-varying features of the diffusion matrix G_t, e.g., its unboundedness at infinity (as in the cognitive model [10]) or its singularity (see the case of diffusion in [11]). In this paper, the average optimal controls over an infinite-time horizon are derived using the extended functional

$$\limsup_{T\to+\infty} \frac{\mathrm{E}J_{2T}(U)}{\int_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}}, \tag{3}$$

which is a generalization of the above-mentioned criteria. In the probabilistic sense, a stronger criterion than the long-run average cost is the pathwise ergodic cost in which the problem

$$\limsup_{T\to+\infty}\{J_{2T}/(2T)\} \to \inf_{U\in\mathcal{U}}$$

is solved with probability 1; see [3]. When considering the impact of the diffusion matrix on the system dynamics, the extended pathwise long-run average cost can be used:

$$\limsup_{T\to+\infty} \frac{J_{2T}(U)}{\int_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}} \quad \text{with probability 1.} \tag{4}$$

Note that the extended long-run average costs were also introduced in [12–14] for the stochastic linear control problem with a one-sided cost functional, i.e., with the limits of integration $[0, T]$ in (2). The problems with two-sided cost functionals were considered in [3, 4]; the criteria of optimality used there are standard for the systems with bounded coefficients (the long-run average and pathwise ergodic costs mentioned above). The admissible controls were assumed to have finite moments of the corresponding processes (more precisely, $\sup_{t\in\mathbb{R}}(\mathrm{E}\|X_t\|^2 + \mathrm{E}\|U_t\|^2) < \infty$ in [4], or $\sup_{t\in\mathbb{R}}(\mathrm{E}\|X_t\|^4 + \mathrm{E}\|U_t\|^4) < \infty$ in [3]), as well as the finite value of the ergodic average $\limsup_{T\to\infty}\{(2T)^{-1}\int_{-T}^{T}\|U_t\|^2\, dt\} < \infty$, in [4]. In comparison with the analysis performed in [3, 4], this paper presents a series of generalizations for the case of finite-dimensional control systems as follows. First, the unbounded time-varying diffusion matrices are allowed ($\|G_t\| \to \infty$, $t \to \pm\infty$), and new extended long-run average cost criteria are considered (see (3) and (4)), which take the former fact into account. Second, problems (3) and (4) are solved for a much wider class of controls than in [3, 4]: we only require the existence of a solution to (1) and the square integrability of controls, i.e., $\int_s^t \|U_\tau\|^2 \, d\tau < \infty$, $-\infty < s \leqslant t < +\infty$. It is important to note the well-known optimal linear feedback law, whose structure also includes the solution of the Riccati equation (for example, see [1, 3, 4]), is also preserved in this case for (3) and (4). As it was established in [3], the optimal path corresponding to this control law has the global asymptotic stability in mean square. In this paper, we obtain more precise estimates for the variations of this process over time, in the mean-square sense and also with probability 1, depending on the coefficients of the diffusion matrix. This seems to be a generalization of the result of [15], where the scalar stationary process was studied. Thus, the aim of this paper is to find an optimal control U_t^* in problems (3) and (4) as well as to analyze the properties of the corresponding optimal path X_t^* as $t \to \pm\infty$. The rest of this paper is organized as follows. In Section 2, basic assumptions on the parameters of the control system (1), (2) are introduced and problem (3) is solved. Section 3 is dedicated to the pathwise optimality of U^* in problem (4) and the stochastic analysis of the path X^*. Besides, in Section 3 some examples of different classes of time-varying diffusion matrices G_t satisfying the basic assumptions are given. In the Conclusions, the outcomes of this paper are outlined and the lines of further research are discussed.

2. AVERAGE OPTIMALITY ON INFINITE HORIZON

First, we formulate the assumptions on the coefficients of (1), (2), which will be used below.

Assumption $\mathcal{AB}$. The pair of matrices (A_t, B_t) is stabilizable for $t \in \mathbb{R}$.

The stabilizability of the pair (A_t, B_t) (e.g., see [2, 4]) means the existence of a bounded piecewise continuous matrix K_t such that the matrix $\mathcal{A}_t = A_t + B_t K_t$, $t \in \mathbb{R}$, is exponentially stable, i.e., the corresponding fundamental matrix $\Phi(t,s)$ admits the upper bound $\|\Phi(t,s)\| \leqslant \kappa_0 e^{-\kappa(t-s)}$, $s \leqslant t$, where $\kappa_0, \kappa > 0$ are constants. It is well-known, that the fundamental matrix is determined by solving the problem $\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s)$, $\Phi(s,s) = I$. The next assumption concerns the disturbance parameters, i.e., the matrix G_t, $t \in \mathbb{R}$. We introduce the set $\mathcal{T} = \{-\infty; +\infty; \pm\infty\}$ and use the compact notation $t \to \mathcal{T}$ for any of the cases $t \to -\infty$, $t \to +\infty$ or $t \to \pm\infty$.

Assumption $\mathcal{G}$. The diffusion matrix G_t satisfies one of the following conditions.

1) G_t is bounded for $t \to \mathcal{T}$.

2) $\|G_t\| \to +\infty$, G_t is differentiable and $d \ln \|G_t\|/dt \to 0$ as $t \to \mathcal{T}$.

It is important to note that the validity of conditions 1 and 2 is related to a particular semi-axis of the parameter $t \in \mathbb{R}$ under consideration (positive or negative). Specifically, for $\|G_t\| = e^{\sqrt[m]{t}}$, where m is an odd number, condition 1 is the case as $t \to -\infty$ while condition 2 as $t \to +\infty$.

According to [2, 4], under Assumption $\mathcal{AB}$ there exists the control law

$$U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*, \tag{5}$$

where a bounded symmetric matrix Π_t is the solution to the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0 \tag{6}$$

and $\Pi_t \geqslant pI$ with a constant $p > 0$. Substituting (5) into (1), we find that the process X_t^*, $t \in \mathbb{R}$, is the solution to the linear stochastic differential equation (SDE)

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dw_t, \tag{7}$$

representing an analog of the Ornstein–Uhlenbeck process for $t \in \mathbb{R}$ in the case of SDEs with time-varying coefficients. Moreover, the matrix $A_t^* = A_t - B_t R_t^{-1} B_t' \Pi_t$ is exponentially stable [2, 4], and some other properties of X_t^*, $t \in \mathbb{R}$, are presented in Lemma 1.

Lemma 1. *Let Assumptions $\mathcal{AB}$ and $\mathcal{G}$ hold. Then the solution to* (7) *is a process of the form* $X_t^* = \int_{-\infty}^t \Phi(t,s)G_s dw_s$, *where* $\Phi(t,s)$ *is the fundamental matrix corresponding to the exponentially stable matrix* $A_t^* = A_t - B_t R_t^{-1} B_t' \Pi_t$. *Moreover, there exists a constant* $c_G > 0$ *such that* $E\|X_t^*\|^2 \leqslant c_G \max\{1, \|G_t\|^2\}$, $t \in \mathbb{R}$.

The proof of Lemma 1 as well as the proofs of all other theoretical results below are postponed to the Appendix. The next theorem establishes the average optimality of the control law U^* over an infinite-time horizon.

Theorem 1. *Let Assumptions $\mathcal{AB}$ and $\mathcal{G}$ hold. Then the control law U^* given by* (5)–(7) *is the solution to the problem*

$$\limsup_{T\to+\infty} \frac{EJ_{2T}(U)}{\int\limits_{-T}^{T} \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}, \tag{8}$$

and

$$0 < \limsup_{T\to+\infty} \frac{EJ_{2T}(U^*)}{\int\limits_{-T}^{T} \|G_t\|^2\,dt} = \limsup_{T\to+\infty} \frac{\int\limits_{-T}^{T} tr(G_t'\Pi_t G_t)\,dt}{\int\limits_{-T}^{T} \|G_t\|^2\,dt} < \infty, \tag{9}$$

where $\mathrm{tr}(\cdot)$ *denotes the matrix trace.*

3. PATHWISE STOCHASTIC OPTIMALITY

Lemmas 2 and 3 characterize the asymptotic properties of the paths of the process X_t^*, $t \in \mathbb{R}$. These properties are necessary to study the stochastic optimality of the control law U^* in problem (4).

Lemma 2. *Let Assumption $\mathcal{AB}$ and item* 2 *of Assumption $\mathcal{G}$ hold. Then there exists a constant* $\bar{c} > 0$ *such that*

$$\limsup_{t\to\mathcal{T}} \frac{\|X_t^*\|^2}{\|G_t\|^2 \ln|t|} < \bar{c} < \infty \quad \textit{with probability } 1,$$

where $|\cdot|$ *denotes the absolute value of a scalar variable.*

The function $h_t = \|G_t\|^2 \ln|t|$ in Lemma 2 is a majorant, i.e., an upper function for the process X_t^* (see [16, Definition 1]) under item 2 of Assumption $\mathcal{G}$. For the bounded diffusion matrix G_t, $t \geqslant 0$, the function h_t was explicitly found in [16]; a special case of a scalar stationary process was considered in [15].

Lemma 3. *Let Assumptions $\mathcal{AB}$ and $\mathcal{G}$ hold. If also* $d\ln\|G_t\|/dt \times \ln|t| \to 0$ *as* $t \to \mathcal{T}$ *in item* 2 *of Assumption $\mathcal{G}$, then*

$$\lim_{T\to+\infty} \frac{\|X_{-T}^*\|^2 + \|X_T^*\|^2}{\int\limits_{-T}^{T} \|G_t\|^2\,dt} = 0 \quad \textit{with probability } 1.$$

The relation in Lemma 3 shows that normalizing previous (X_{-T}^*) and subsequent (X_T^*) values of the path by $\Gamma_T = \sqrt{\int_{-T}^T \|G_t\|^2\,dt}$, as a result, we obtain a vanishing process a.s. when the observation "window" increases. The function Γ_T defined in this way determines the standard deviation of the integrated vector disturbances over the period $[-T, T]$; more specifically, $\mathcal{Z}_T = \int_{-T}^{T} G_t dw_t$ and then $\mathrm{E}\|\mathcal{Z}_T\|^2 = \int_{-T}^{T} \|G_t\|^2\, dt$.

To analyze problem (4) in the case $\|G_t\| \to \infty$, $t \to \mathcal{T}$, we need a stronger condition than item 2 of Assumption $\mathcal{G}$.

Assumption $\mathcal{G}1$. Let item 2 of Assumption $\mathcal{G}$ hold and additionally, $d \ln \|G_t\|/dt \times \ln |t|(\ln\ln |t| + \ln\ln \|G_t\|) \to 0$, $t \to \mathcal{T}$, where $\|G_t\|$ is a monotone function as $t \to \mathcal{T}$.

The main result of this section is Theorem 2 on the pathwise optimality of the control law U^*.

Theorem 2. *Let the hypotheses of Theorem 1 and Assumption* $\mathcal{G}1$ *hold. If* $\int_{-T}^{T} \|G_t\|^2\, dt \to \infty$ *as* $T \to +\infty$, *then the average optimal control law* U^* *is also the solution to the control problem with the extended pathwise long-run average cost, i.e.,*

$$\limsup_{T\to+\infty} \frac{J_{2T}(U)}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}} \quad \textit{with probability } 1, \tag{10}$$

and

$$\limsup_{T\to+\infty} \frac{J_{2T}(U)}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} = \limsup_{T\to+\infty} \frac{EJ_{2T}(U)}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} \qquad \textit{a.s.} \tag{11}$$

We give some examples of various classes of functions describing the dynamics of the diffusion matrix norm G_t. For two scalar nonnegative functions f_t and g_t, the relation $f_t \sim g_t$ means that $0 < \lim_{t\to\pm\infty}(f_t/g_t) < \infty$.

Example 1.

1. The power family $\|G_t\|^2 \sim |t|^{2\alpha}$, $\alpha \in \mathbb{R}$: for $\alpha \leqslant 0$, item 1 of Assumption $\mathcal{G}$ holds; for $\alpha > 0$, item 2. Since $d \ln \|G_t\|/dt \sim 1/|t|$ and $\ln\ln \|G_t\| \sim \ln |t|$, the Assumption $\mathcal{G}1$ is valid for any α. In addition, the hypotheses of Theorem 2 are satisfied for $\alpha \geqslant -1/2$.

2. The logarithmic family $\|G_t\|^2 \sim \ln^{2\alpha} |t|$, $\beta \in \mathbb{R}$: if $\beta \leqslant 0$, then item 1 of Assumption $\mathcal{G}$ holds; if $\beta > 0$, item 2. Due to $d \ln \|G_t\|/dt \sim 1/(|t| \ln |t|)$ and $\ln\ln \|G_t\| \sim \ln\ln |t|$, Assumption $\mathcal{G}1$ is true for any β. In addition, for each $\beta \in \mathbb{R}$ the hypotheses of Theorem 2 are satisfied.

3. The exponential family $\|G_t\|^2 \sim e^{|t|^\mu}$, $\mu < 1$: for $\mu \leqslant 0$, item 1 of Assumption $\mathcal{G}$ holds; for $\mu > 0$, item 2. Also, $d \ln \|G_t\|/dt \sim |t|^{\mu-1}$ and $\ln\ln \|G_t\| \sim |t|^\mu$, i.e., the relation from Assumption $\mathcal{G}1$ follows for any $0 < \mu < 1$. Obviously, the hypotheses of Theorem 2 are satisfied for each $\mu < 1$.

4. CONCLUSIONS

In this paper, the stochastic linear control problem over an infinite-time horizon with a two-sided cost functional and a time-varying diffusion matrix G_t has been considered. In the two-sided quadratic cost functional $J_{2T}(U)$ (2), the limits of integration have opposite sign and depend on the length of planning horizon, i.e., $t \in [-T, T]$ in (2) and then $T \to +\infty$. Under the standard stabilizability condition of the deterministic system (see Assumption $\mathcal{AB}$) and conditions on the diffusion matrix growth (see Assumptions $\mathcal{G}$ and $\mathcal{G}1$), it has been shown that the well-known linear feedback law U^* (5)–(7) is optimal with respect to the extended generalized long-run average cost (Theorem 1) and its pathwise analog (Theorem 2). Also, the asymptotic probabilistic behavior of X_t^*—the optimal path (7) of the system—has been studied. In particular, it has been established that the upper estimates on variations of X_t^* in the mean-square sense can be determined depending on $\|G_t\|$ (Lemma 1). In the pathwise dynamics, a sufficient normalization under which the process tends to zero with probability 1 has been found (Lemma 3); this normalization has been defined via a statistical characteristic (standard deviation) of the integrated vector disturbances. Concerning the lines of further research, we mention the stochastic path tracking problem in a more general setup than, e.g., the model [7], where the reference path was a Gaussian process.

APPENDIX

Proof of Lemma 1. Since $X_t^* = \Phi(t,0)\chi_t$, where $\chi_t = \int_{-\infty}^t \Phi(0,s)G_s dw_s$, first we have to show the existence of the stochastic integral χ_t with infinite lower limit; then, using differentiation we have to check that X_t^* satisfies (7). Due to the definition of a two-sided Wiener process $w_t = w_{-t}^{(2)}$, $t<0$, where $w_\tau^{(2)}$ is a standard Wiener process and $\tau \geqslant 0$, the stochastic calculus of integrals χ_t obeys the same rules as Itô integration; also, see [6, pp. 13–14]. For $t \geqslant 0$, the process $X_t^* = \Phi(t,0)X_0^* + \int_0^t \Phi(t,s)G_s dw_s$, where $X_0^* = \chi_0$. It is well-known [6, Theorem 5.1, p. 54], that the existence of χ_t, $t\in\mathbb{R}$, is related to the condition $\mathrm{E}\|\chi_0\|^2 = \int_{-\infty}^0 \|\Phi(0,s)G_s\|^2\,ds < \infty$, which holds because of the exponential stability of the matrix A_t^* and Assumption $\mathcal{G}$. Indeed, $\|\Phi(0,s)\| \leqslant \kappa_0 e^{\kappa s}$, $s\leqslant 0$, and $\limsup_{s\to-\infty}\|G_s\|^2 e^{\gamma s} < \infty$ for any $\gamma>0$. Then choosing $\gamma < 2\kappa$ gives $\mathrm{E}\|\chi_0\|^2<\infty$. Next, we obtain

$$\mathrm{E}\|X_t^*\|^2 = \int_{-\infty}^{t} \mathrm{tr}\{\Phi(t,s)G_sG_s'\Phi'(t,s)\}\,ds \leqslant c\int_{-\infty}^{t} e^{-2\kappa(t-s)}\|G_s\|^2\,ds, \tag{A.1}$$

where $\mathrm{tr}(\cdot)$ denotes the matrix trace; hereinafter, c is some positive constant whose precise value makes little sense and may vary from formula to formula. From (A.1) it follows that for a bounded diffusion matrix G_t, the expression of $\mathrm{E}\|X_t^*\|^2$, $t\in\mathbb{R}$, is bounded as well. If $\|G_t\|\to+\infty$, $t\to\mathcal{T}$, then integration by parts (like in [14, Lemma 1] for the case $t\to+\infty$) can be used for showing that $\limsup_{t\to\mathcal{T}}(\mathrm{E}\|X_t^*\|^2/\|G_t\|^2)<\infty$. The proof of Lemma 1 is complete.

Proof of Theorem 1. We fix a control $U\in\mathcal{U}$ and determine the corresponding process (1). Let $x_t = X_t^* - X_t$, $u_t = U_t^* - U_t$, and $\bar{x} = X_0^* - X_0$. Then we have the representation

$$J_{2T}(U^*) - J_{2T}(U) = 2x_T'\Pi_T X_T^* - 2x_{-T}'\Pi_{-T}X_{-T}^* - \int_{-T}^{T}(x_t'Q_tx_t + u_t'R_tu_t)dt - 2\int_{-T}^{T}x_t'\Pi_tG_tdw_t. \tag{A.2}$$

For estimating (A.2), the dynamics of x_t with $t\in[-T,T]$ are analyzed. By construction,

$$dx_t = A_tx_tdt + B_tu_tdt. \tag{A.3}$$

First, let $t\in[0,T]$. In this case, the consideration of (A.3) with the initial condition $x_0=\bar{x}$ under the assumption $Q_t\geqslant qI$ yields a solution to (A.3) of the form $x_T = \bar\Phi(T,0)\bar{x} + \int_0^T \bar\Phi(T,t)\times(\bar{k}\sqrt{Q_t}x_t + B_tu_t)dt$, where $\bar\Phi(t,s)$ is the fundamental matrix that corresponds to the exponentially stable matrix $\bar{A}_t = A_t - \bar{k}\sqrt{Q_t}$ for some constant $\bar{k}>0$. This relation can be estimated as

$$\|x_T\|^2 \leqslant \bar{c}e^{-\bar\kappa T}\|\bar{x}\|^2 + \bar{c}\int_0^T e^{-\bar\kappa(T-s)}(x_s'Q_sx_s + u_s'R_su_s)ds, \tag{A.4}$$

where $\bar{c},\bar\kappa>0$ are some constants. In the case $t\in[-T,0]$, Eq. (A.3) is considered with the boundary condition $x_0=\bar{x}$. Due to $Q_t\geqslant qI$, there exists a constant $\tilde{k}>0$ such that the matrix $\tilde{A}_t = A_t + \tilde{k}\sqrt{Q_t}$ is exponentially antistable, i.e., $\|\tilde\Phi(s,t)\|\leqslant\tilde\kappa e^{-\tilde\kappa_1(t-s)}$, $s\leqslant t$, where $\tilde\kappa,\tilde\kappa_1>0$

are constants. Then, writing the solution to (A.3) in the form $x_{-T} = \tilde{\Phi}(-T,0)\bar{x} - \int_{-T}^{0}\tilde{\Phi}(-T,s)\times(\tilde{k}\sqrt{Q_s}x_s + B_s u_s)ds$, we obtain the upper bound

$$\|x_{-T}\|^2 \leqslant \tilde{c}e^{-\tilde{\kappa}_1 T}\|\bar{x}\|^2 + \tilde{c}\int\limits_{-T}^{0} e^{-\tilde{\kappa}_1(T+s)}(x_s' Q_s x_s + u_s' R_t u_s)ds \qquad \text{(A.5)}$$

with some constant $\tilde{c} > 0$. Consequently, the boundedness of Π_t, $t \in \mathbb{R}$, together with the elementary inequality $2ab \leqslant ca^2 + b^2/c$, holding for an arbitrary constant $c > 0$, and (A.4), (A.5) lead to the following upper bound on the average value of (A.2):

$$\mathrm{E}J_{2T}(U^*) - \mathrm{E}J_{2T}(U) \leqslant c_0 e^{-\kappa_1}\|\bar{x}\|^2 + c_1\mathrm{E}\|X_T^*\|^2 + c_2\mathrm{E}\|X_{-T}^*\|^2,$$

with some constants $\kappa_1, c_0, c_1, c_2 > 0$. Next, normalizing by $\int_{-T}^{T}\|G_t\|^2$ in view of Lemma 1 and the conditions of Assumption $\mathcal{G}$, in the the limit as $T \to +\infty$ we arrive at the relation

$$\limsup_{T\to+\infty}\frac{\mathrm{E}J_{2T}(U^*)}{\int\limits_{-T}^{T}\|G_t\|^2\,dt} \leqslant \limsup_{T\to+\infty}\frac{\mathrm{E}J_{2T}(U)}{\int\limits_{-T}^{T}\|G_t\|^2\,dt},$$

which proves that U^* is the solution to Problem (3). Note that for the processes defined by (7) for all $t \in \mathbb{R}$, the solution to the corresponding equation has the integral form $X_t^* = X_s^* + \int_s^t A_\tau^* X_\tau^* d\tau + \int_s^t G_\tau\,dw_\tau$ for an arbitrary $s \in \mathbb{R}$, $s \leqslant t$. In accordance with [17, Remark 4.3.7, p. 99], the well-known results (in particular, the Itô formula) can be applied to such processes. By the Itô formula,

$$J_{2T}(U^*) = [(X_{-T}^*)'\Pi_{-T}X_{-T}^*] - [(X_T^*)'\Pi_T X_T^*] + \int\limits_{-T}^{T}\mathrm{tr}(G_t'\Pi_t G_t)\,dt + 2\int\limits_{-T}^{T}(X_t^*)'\Pi_t G_t dw_t. \qquad \text{(A.6)}$$

Based on the inequality from Lemma 1 and the property $pI \leqslant \Pi_t \leqslant cI$, $t \in \mathbb{R}$, for the average value of (A.6) we write the two-sided estimate $\hat{c}_1\int_{-T}^{T}\|G_t\|^2\,dt \leqslant \mathrm{E}J_{2T}(U^*) \leqslant \hat{c}_2\int_{-T}^{T}\|G_t\|^2\,dt$ with some constants $\hat{c}_1, \hat{c}_2 > 0$, which finally gives (9). The proof of Theorem 1 is complete.

Proof of Lemma 2. In the case $\mathcal{T} = +\infty$, $X_t^* = \Phi(t,0)X_0^* + \tilde{X}_t^*$, where $\tilde{X}_t^* = \int_0^t\Phi(t,s)G_s dw_s$, $t \geqslant 0$. In [14, Lemma 2] it was shown that $\|\tilde{X}_t^*\|^2 \leqslant c_0\|G_t\|^2\ln t$ a.s. as $t \to +\infty$, where $c_0 > 0$ is a deterministic constant. Since X_0^* is a random variable and $\|\Phi(t,0\| \leqslant \kappa_0 e^{-\kappa t}$, the result for $\|\tilde{X}_t^*\|^2$ given above implies the desired result. In the case $\mathcal{T} = -\infty$, first consider the scalar process z_t with the dynamics $dz_t = -\kappa z_t dt + \sigma_t dw_t$, $\kappa > 0$, and the diffusion coefficient σ_t, which satisfies the hypotheses of Lemma 2. Then $z_t = e^{-\kappa t}I_t$, where $I_t = \int_{-\infty}^{t}e^{\kappa s}\sigma_s dw_s$. In the stochastic integral I_{-T}, $T \geqslant 0$, we may perform the change of time $\tau = -1/s$, taking into account that $\tau w_{-1/\tau} = \hat{w}_\tau$, where $\hat{w}_\tau$, $\tau \geqslant 0$, is another Wiener process; for example, see [18, p. 94]. Therefore,

$$I_{-T} = \int\limits_{0}^{1/T} e^{-\kappa/\tau}\sigma_{-1/\tau}\left(\frac{d\hat{w}_\tau}{\tau} - \frac{\hat{w}_\tau}{\tau^2}d\tau\right).$$

As $T \to +\infty$, the terms in I_{-T} can be estimated using the local law of the iterated algorithm [18, Corollary 3, p. 93]. Let $I_T^{(1)} = \int_0^{1/T}e^{-\kappa/\tau}\sigma_{-1/\tau}\frac{d\hat{w}_\tau}{\tau}$; then $|I_T^{(1)}| \leqslant \hat{c}_1 h_T^{(1)}$ for $h_T^{(1)} = \sqrt{M_T\ln\ln(1/M_T)}$, $M_T = \int_0^{1/T}e^{-2\kappa/\tau}\sigma_{-1/\tau}^2\frac{d\tau}{\tau^2}$ and some constant $\hat{c}_1 > 0$. As $T \to +\infty$, the process $I_T^{(2)} = \int_0^{1/T}e^{-\kappa/\tau}\times\sigma_{-1/\tau}\frac{\hat{w}_\tau}{\tau^2}d\tau$ admits the upper bound $|I_T^{(2)}| \leqslant \hat{c}_2 h_T^{(2)}$, where $h_T^{(2)} = \int_0^{1/T}\frac{e^{-\kappa/\tau}\sqrt{\tau\ln\ln(1/\tau)}}{\tau^2}|\sigma_{-1/\tau}|d\tau$ and $\hat{c}_2 > 0$ is some constant. With l'Hôpital's rule, it is easy to demonstrate that

$$\left(h_T^{(1)} + h_T^{(2)}\right)/\sqrt{\left(e^{2\kappa T}\sigma_{-T}^2\ln T\right)} \to c, \quad T \to +\infty.$$

Then $\limsup_{T\to+\infty}\{|z_T|/\sqrt{(\sigma_{-T}^2 \ln T)}\} < \infty$, this relation used to estimate each component of the auxiliary process $\hat{X}_{-T} = \int_{-\infty}^{-T} e^{\kappa(T+s)} G_s dw_s$ guarantees the existence of a constant $\hat{c} > 0$ such that $\limsup_{T\to+\infty}\{\|\hat{X}_{-T}\|/h_T\} < \hat{c} < \infty$ a.s. if $h_T = \sqrt{\|G_{-T}\|^2 \ln T}$. Next, for the difference $Z_t = X_t^* - \hat{X}_t$ with the dynamics $dZ_t = A_t^* Z_t dt + (\kappa I - A_t^*)\hat{X}_t dt$ and the solution $Z_t = \int_{-\infty}^{t} \Phi(t,s)(\kappa I - A_s^*)\hat{X}_s\, ds$, the exponential stability of A_t^* and $\dot{h}_t/h_t \to 0$ imply that the ratio $\|Z_t\|/h_t$ is bounded as $t \to -\infty$; this fact can be established by a standard approach (e.g., see [16]). As a result, $\limsup_{t\to-\infty}\{\|X_t^*\|/h_t\} < \bar{c} < \infty$ for $h_t = \sqrt{\|G_t\|^2 \ln|t|}$. The proof of Lemma 2 is complete.

Proof of Lemma 3. Under item 2 of Assumption $\mathcal{G}$, Lemma 2 together with the condition $d\ln\|G_t\|/dt \ln|t| \to 0$, $t \to \mathcal{T}$, lead to

$$\lim_{t\to\mathcal{T}} \left\{ \|X_t^*\|^2 \Big/ \left| \int_0^t \|G_s\|^2\, ds \right| \right\} \leqslant c \lim_{t\to\mathcal{T}} \left\{ \|G_t\|^2 \ln|t| \Big/ \left| \int_0^t \|G_s\|^2\, ds \right| \right\} = 0 \quad \text{with probability 1.}$$

If the diffusion matrix G_t is bounded, then for $\mathcal{T} = +\infty$ we again adopt the representation $X_T^* = \Phi(T,0)X_0^* + \tilde{X}_T^*$, where $\tilde{X}_T^* = \int_0^T \Phi(T,s)G_s dw_s$, $T \geqslant 0$, and the well-known result [13, Theorem 1], stating that $\|\tilde{X}_T^*\|^2/\int_0^T \|G_s\|^2 ds \to 0$ a.s. as $T \to +\infty$. Then, in view of the decreasing exponential upper bound on $\|\Phi(T,0)\|$, we obtain the relation $\|X_T^*\|^2/\int_{-T}^{T}\|G_s\|^2 ds \to 0$ as $T \to +\infty$. For $\mathcal{T} = -\infty$, due to the representation $\|X_{-T}^*\|^2 = \|X_0^*\|^2 - \int_{-T}^0 (X_t^*)'(A_t + A_t')X_t^* dt - \int_{-T}^0 (X_t^*)' G_t dw_t - \int_{-T}^0 (dw_t)' G_t' X_t^* - \int_{-T}^0 \|G_t\|^2\, dt$ the terms can be analyzed using [13, Lemma 1, Lemma 2] with the change of time $\tau = -t$ in the integrand; as a result, $\|X_{-T}^*\|^2/\left|\int_{-T}^0 \|G_s\|^2\, ds\right| \to 0$ as $T \to +\infty$. The proof of Lemma 3 is complete.

Proof of Theorem 2. In order to estimate (A.2), we use inequalities (A.4) and (A.5). Replacing T by t in (A.4) and $(-T)$ by t in (A.5) and integrating the resulting expressions on $[0,T]$ and $[-T,0]$, respectively, yield

$$\int_0^T \|x_t\|^2\, dt \leqslant \bar{c}_1\|\bar{x}\|^2 + \bar{c}_1 \int_0^T (x_t' Q_t x_t + u_t' R_t u_t) dt \tag{A.7}$$

and

$$\int_{-T}^0 \|x_t\|^2\, dt \leqslant \tilde{c}_1\|\bar{x}\|^2 + \tilde{c}_1 \int_{-T}^0 (x_t' Q_t x_t + u_t' R_t u_t) dt \tag{A.8}$$

with some constants $\bar{c}_1, \tilde{c}_1 > 0$. Then (A.2) can be estimated as

$$J_{2T}(U^*) \leqslant J_{2T}(U) + c_0\|\bar{x}\|^2 + c_1\|X_T^*\|^2 + c_2\|X_{-T}^*\|^2 - c_3\int_{-T}^T \|x_t\|^2\, dt - 2\int_{-T}^T x_t'\Pi_t G_t dw_t,$$

where $c_0, c_1, c_2, c_3 > 0$ are some constants. Consequently,

$$J_{2T}(U^*) \leqslant J_{2T}(U) + \mathcal{R}_T^{(0)} + \mathcal{R}_T^{(+)} + \mathcal{R}_T^{(-)}, \tag{A.9}$$

where the processes are

$$\mathcal{R}_T^{(0)} = c_0\|\bar{x}\|^2 + c_1\|X_T^*\|^2 + c_2\|X_{-T}^*\|^2,$$

$$\mathcal{R}_T^{(+)} = -c_3\int_0^T \|x_t\|^2\, dt - 2\int_0^T x_t'\Pi_t G_t dw_t, \quad \text{and} \quad \mathcal{R}_T^{(-)} = -\mathcal{R}_{-T}^{(+)}.$$

Recall that Assumptions $\mathcal{G}$ and $\mathcal{G}1$ are satisfied and also $\int_{-T}^{T} \|G_t\|^2\,dt \to \infty$ as $T \to +\infty$; hence, by Lemma 3,

$$\lim_{T\to+\infty} \left\{ \mathcal{R}_T^{(0)} \Big/ \int_{-T}^{T} \|G_t\|^2\,dt \right\} = 0 \quad \text{a.s.}$$

Next, consider the behavior of the processes $\mathcal{R}_T^{(+)}$ and $\mathcal{R}_T^{(-)}$ as $T \to +\infty$. As is well-known (e.g., see [12]), for a bounded diffusion matrix G_t, $t \geqslant 0$, the inequality $\limsup\limits_{t\to+\infty}\{\mathcal{R}_T^{(+)}/g_T\} \leqslant 0$ holds a.s. for any function $g_T > 0$ such that $g_T \to \infty$, $t \to +\infty$. By the hypothesis $g_T = \int_{-T}^{T} \|G_t\|^2\,dt$ can be taken as the normalizing function. If item 2 of Assumption $\mathcal{G}$ and also Assumption $\mathcal{G}1$ hold, then using the law of the iterated algorithm of stochastic integrals (e.g., see [19]), $\mathcal{R}_T^{(+)}$ can be estimated as $|\mathcal{R}_T^{(+)}| \leqslant L_T,\ T \to +\infty$, where

$$L_T = \hat{c}_1 \|G_T\|^2 \sqrt{\int_0^T \|x_t\|^2\,dt \ln\ln\left(\int_0^T \|x_t\|^2\,dt\right)} - \hat{c}_2 \int_0^T \|x_t\|^2 dt + \hat{c}_3 \|G_T\|^2 \ln\ln\|G_T\|,$$

and $\hat{c}_1, \hat{c}_2, \hat{c}_3$ are some constants. Using the same arguments as in the proof of [14, Lemma 3], we establish the inequality $L_T \leqslant c\|G_T\|^2 \ln\ln\|G_T\|$, and consequently $\limsup\limits_{t\to+\infty}\{\mathcal{R}_T^{(+)}/g_T\} = 0$ a.s. for $g_T = \|G_T\|^2 \ln\ln\|G_T\|$. From this result and Assumption $\mathcal{G}1$ it follows that $\lim\limits_{T\to+\infty}\left\{\dfrac{g_T}{\int_{-T}^{T}\|G_t\|^2 dt}\right\} = 0$ a.s. Also note that the results on the choice of the normalizing functions g_T for the process $\mathcal{R}_T^{(-)}$ are obtained using the relations for $\mathcal{R}_T^{(+)}$ (see above) with the change of variable $\tau = -t$ in the integrands. Due to these remarks, the passage to the limit as $T \to +\infty$ for (A.9) gives the inequality

$$\limsup_{T\to+\infty} \frac{J_{2T}(U^*)}{\int_{-T}^{T} \|G_t\|^2\,dt} \leqslant \limsup_{T\to+\infty} \frac{J_{2T}(U)}{\int_{-T}^{T} \|G_t\|^2\,dt} \qquad \text{with probability 1.}$$

Next, consider (A.6); in view of (9) and Lemma 3, to prove (11) we have to study the behavior of

$$I_T = \int_{-T}^{T} (X_t^*)' \Pi_t G_t\,dw_t = I_T^{(+)} + I_T^{(-)},$$

where $I_T^{(+)} = \int_0^T (X_t^*)'\Pi_t G_t\,dw_t$, $I_T^{(-)} = -I_{-T}^{(+)}$. More specifically, it is necessary to analyze $I_T^{(+)}/\Gamma_T$, with $\Gamma_T = \int_0^T \|G_t\|^2\,dt$; note that the case $I_T^{(-)}/|\Gamma_{-T}|$ is treated similarly through the change of time. For a bounded diffusion matrix G_t, $t \geqslant 0$, the ratio $I_T^{(+)}/\,\Gamma_T \to 0$ a.s. as $T \to +\infty$; see [13]. For $\|G_t\| \to \infty$, $t \to +\infty$, and the relations of Assumption $\mathcal{G}1$, we apply the law of the iterated algorithm for stochastic integrals [19], which claims that $\limsup\limits_{T\to+\infty}\left\{|I_T^{(+)}|/\sqrt{\langle I_T^{(+)}\rangle \ln\ln\langle I_T^{(+)}\rangle}\right\} < \infty$ a.s., where $\langle I_T^{(+)}\rangle = \int_0^T \|X_t^*\|^2 \|G_t\|^2 \|\Pi_t\|^2 dt$. Lemma 2 together with the monotonicity property of $\|G_t\|$ yields the upper bounds $\langle I_T^{(+)}\rangle \leqslant c\|G_T\|^2 \int_0^T \|G_t\|^2 dt \ln T$ and $\ln\ln\langle I_T^{(+)}\rangle \leqslant c(\ln\ln T + \ln\ln\|G_T\|)$. Then

$$\langle I_T^{(+)}\rangle \ln\ln\langle I_T^{(+)}\rangle \,/\, \Gamma_T^2 \leqslant c\|G_T\|^2\,(\ln\ln T + \ln\ln\|G_T\|)\ln T \,/\, \Gamma_T \to 0, \quad T \to +\infty.$$

(Here convergence to 0 follows from Assumption $\mathcal{G}1$.) Consequently, $I_T^{(+)}/\Gamma_T \to 0$ with probability 1. In accordance with the aforesaid,

$$I_T/\int_{-T}^{T} \|G_t\|^2\,dt \to 0 \quad \text{a.s.,} \quad T \to +\infty,$$

and (11) holds. The proof of Theorem 2 is complete.

ACKNOWLEDGMENTS

This work was performed within the research project "Theory and Methods of Computer Simulation, Mathematical Modeling and Analysis of Social Systems and Processes," state registration no. AAAA-F18-118021990120-2.

REFERENCES

1. Kwakernaak, H. and Sivan, R., *Linear Optimal Control Systems*, Hoboken: Wiley-Interscience, 1972.
2. Mueller, M. and Cantoni, M., Normalized Coprime Representations for Time-Varying Linear Systems, *Proc. 49th IEEE Conf. on Decision and Control*, New York, 2010, pp. 7718–7723.
3. Tudor, C., Quadratic Control for Linear Stochastic Equations with Pathwise Cost, *Stochastic Systems and Optimization, Proc. 6th IFIP WG 7.1 Working Conf.*, Warsaw, Poland, September 12–16, 1988, Berlin: Springer, 1989, pp. 360–369.
4. Da Prato, G. and Ichikawa, A., Quadratic Control for Linear Time-Varying Systems, *SIAM J. Control Optim.*, 1990, vol. 28, no. 2, pp. 359–381.
5. Makila, P.M., Convoluted Double Trouble, *IEEE Control Syst. Mag.*, 2002, vol. 22, no. 4, pp. 26–31.
6. Nourdin, I., *Selected Aspects of Fractional Brownian Motion*, Milan: Springer, 2012.
7. Altman, E., Basar, T., and Hovakimyan, N., Worst-Case Rate-Based Flow Control with an ARMA Model of the Available Bandwidth, in *Advances in Dynamic Games and Applications*, Boston: Birkhauser, 2000, pp. 3–29.
8. Sun, T. and Nielsen, S.R.K., Stochastic Optimal Control of a Heave Point Wave Energy Converter Based on a Modified LQG Approach, *Ocean Eng.*, 2018, vol. 154, pp. 357–366.
9. Grimble, M.J. and Johnson, M.A., *Optimal Control and Stochastic Estimation: Theory and Applications*, New York: Wiley, 1986, vol. 2.
10. Smith, P.L. and McKenzie, C.R.L., Diffusive Information Accumulation by Minimal Recurrent Neural Models of Decision Making, *Neural Comput.*, 2011, vol. 23, no. 8, pp. 2000–2031.
11. Lim, S.C. and Muniandy, S.V., Self-Similar Gaussian Processes for Modeling Anomalous Diffusion, *Phys. Rev. E*, 2002, vol. 66, no. 2, p. 021114.
12. Belkina, T.A. and Palamarchuk, E.S., On Stochastic Optimality for a Linear Controller with Attenuating Disturbances, *Autom. Remote Control*, 2013, vol. 74, no. 4, pp. 628–641.
13. Palamarchuk, E.S., Asymptotic Behavior of the Solution to a Linear Stochastic Differential Equation and Almost Sure Optimality for a Controlled Stochastic Process, *Comput. Math. Math. Phys.*, 2014, vol. 54, no. 1, pp. 83–96.
14. Palamarchuk, E.S., Risk Estimation in Linear Economic Systems under Negative Time Preferences, *Ekon. Mat. Met.*, 2013, vol. 49, no. 3, pp. 99–116.
15. Al-Azzawi, S., Liu, J., and Liu, X., Convergence Rate of Synchronization of Systems with Additive Noise, *Discrete Contin. Dyn. Syst., Ser. B*, 2017, vol. 22, no. 2, pp. 227–245.

16. Palamarchuk, E.S., On the Generalization of Logarithmic Upper Function for Solution of a Linear Stochastic Differential Equation with a Nonexponentially Stable Matrix, *Differ. Equat.*, 2018, vol. 54, no. 2, pp. 193–200.

17. Prevot, C. and Rockner, M., *A Concise Course on Stochastic Partial Differential Equations*, Berlin: Springer, 2007.

18. Bulinskii, A.V. and Shiryaev, A.N., *Teoriya sluchainykh protsessov* (Theory of Random Processes), Moscow: Fizmatlit, 2005.

19. Wang, J., A Law of the Iterated Logarithm for Stochastic Integrals, *Stoch. Proc. Appl.*, 1993, vol. 47, no. 2, pp. 215–228.

This paper was recommended for publication by B.M. Miller, a member of the Editorial Board