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Analysis of Criteria for Long-run Average in the Problem of Stochastic Linear Regulator

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Abstract—The optimality criteria used in the problem of stochastic linear regulator over an infinite time horizon were analyzed. A certain criterion for long-run average and pathwise ergodic were shown to be inefficient with regard for the disturbance factor. Consideration was given to a new criterion of the extended long-run average and its use in the discounted control systems.

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1. INTRODUCTION: MODEL DESCRIPTION AND PROBLEM OF CHOOSING THE OPTIMALITY CRITERION

In the control problems, choice of the optimality criterion is one of the most important questions. For the stochastic systems, the expectation of the objective functional is usually carried out over a finite horizon. If with increase of the planning interval the expected value of the functional also becomes unbounded, then at formulating the problem over an infinite time horizon the criterion of long-run average is often used; see, for example, [1]. The present paper considers a control system known as the stochastic linear regulator. Let then on the complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ given be an n-dimensional stochastic process $X_t,\ t \geqslant 0$, obeying the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dw_t, \qquad X_0 = x, \tag{1}$$

where the initial state x is a non-random; w_t, $t \geqslant 0$, is the standard d-dimensional Wiener process; U_t, $t \geqslant 0$, is an admissible control or k-dimensional stochastic process adapted to filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_t = \sigma\{w_s,\ s \leqslant t\}$ such that Eq. (1) admits a solution; $A_t,\ B_t,\ G_t,\ t \geqslant 0$, are bounded matrices having dimensions for which (1) has sense, $\int_0^\infty \|G_t\|^2\, dt > 0$. Denote by $\mathcal{U}$ the set of admissible controls.

Performance of the control used over the interval $[0, T]$ is estimated by the quadratic objective functional

$$J_T(U) = \int\limits_0^T (X_t' Q_t X_t + U_t' R_t U_t)\, dt, \tag{2}$$

where U_t is the admissible control $U \in \mathcal{U}$ over the interval $[0, T]$; $Q_t \geqslant 0$, $R_t \geqslant \rho I$, $t \geqslant 0$, are bounded symmetrical matrices, where $'$ is the transpose sign, $\rho > 0$ is a constant, the notation $A \geqslant B$ means that the difference of the matrices $A - B$ is positive semidefinite, and I is the identity matrix.

Then the control problem for (1), (2) with the criterion of long-run average is given by

$$\limsup_{T\to\infty} \frac{\mathrm{E} J_T(U)}{T} \to \inf_{U\in\mathcal{U}}. \tag{3}$$

As was noted in [2], the long-run average (3) disregards the fundamental factor of the integral noise actions $\mathcal{Z}_T = \int_0^T G_t\,dw_t$ affecting the system behavior over a long-run period. As the result, it was indicated to possible inefficiency of this criterion, for example, in the case of damped perturbations, that is, for $\lim_{t\to\infty} \|G_t\| = 0$, as well as to its inapplicability to the discounted problems where in (2) the matrices $Q_t = f_t Q$ and $R_t = f_t R$, where $f_t > 0$ is a decreasing discount function. As the result, the notion of extended long-run average using in the normalization $\mathrm{E}(\mathcal{Z}_T'\mathcal{Z}_T) = \int_0^T \|G_t\|^2\,dt$ as the sum of variances of the components of the vector $\mathcal{Z}_T$ was introduced, and consideration was given to the problem of control like

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}. \tag{4}$$

It turned out (see [2]) that under natural assumptions on the system parameters the well-known stable control law U^* established as the limit (for $T \to \infty$) of solutions U^{*T} of the problems of control over finite intervals is optimal for the new criterion. Additionally, if $\int_0^T \|G_t\|^2\,dt \to \infty$, $T \to \infty$, then, as it was shown in [3], that U^* also becomes solution of the problem with a pathwise analog of the extended long-run average:

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int\limits_0^T \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \text{ with probability one.} \tag{5}$$

The criterion in (5) represents a generalization of the pathwise long-run average $\limsup\limits_{T\to\infty} (J_T(U)/T)$ called also the pathwise ergodic average, see [1]. The choice of normalization $\int_0^T \|G_t\|^2 dt$ in (4) and (5) is due to the fact that $\mathrm{E}J_T(U^*) = \int_0^T \mathrm{tr}(G_t'\Pi_t G_t)\,dt + l_T$, where $\mathrm{tr}(\cdot)$ is the matrix trace, l_t is a bounded function, Π_t is a positive semidefinite bounded matrix [2], and $J_T(U^*) = \mathrm{E}J_T(U^*) + \xi_T$ and $\lim\limits_{T\to\infty} (\xi_T / \int_0^T \|G_t\|^2\,dt) = \xi$ almost sure (a.s.), where ξ is a random variable (see [3]). At the same time, as will be shown below, the criteria of long-run and pathwise long-run average can take identical values over a whole set of controls that are not somehow related with the property of optimality under a finite T.

Practical application of the results obtained includes possible use of the extended long-run averages for control of systems where the specificity of the considered plants presumes time-dependence of the parameters of their dynamics, including varying nature of the perturbation variance. Motion control [4, 5], flow control in the wireless networks [6, 7], problems in mechanics [8] and neurobiology [9] exemplify such formulations. In addition, analysis of the performance of the control actions may require estimation of the functional on the controls close to the optimal (see, for example, [10] for the case of constant parameters) which gives rise to the need for certain studies to be carried out here in what follows. The problems of long-run stabilization of the economic systems with a quite general discounted [11] such as the macroeconomic control [12] can be reckoned among another application class.

The present paper aims at investigating efficiency of the criterion for long-run average, its generalization, as well as their pathwise versions in the stochastic system (1), (2) from the standpoint of most precise allowance for the order of functional variation over the stable optimal control law. Section 2 introduces the notion of criterion efficiency, formulates assumptions about the system parameters, as well as auxiliary assertions about the asymptotic behavior of the integral quadratic functionals of the solutions of the linear stochastic differential equations. Section 3 presents the main results obtained in the paper and an example of analyzing the criteria for the case of scalar linear regulator. Section 4 discusses application of the extended long-run average to the problems of discount system control. Conclusions summarize the study, and all statements are proved in the Appendix.

2. BASIC ASSUMPTIONS AND AUXILIARY RESULTS

We first formulate the main assumption about the parameters of system (1)–(2) enabling one to study the control problem under $T \to \infty$.

Assumption $\mathcal{P}$. *The functions A_t, B_t, Q_t, and R_t, $t \geqslant 0$, are such that there exists an absolutely continuous bounded function Π_t, $t \geqslant 0$ with values in the set of positive semidefinite symmetric matrices satisfying the Riccati equation*

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0 \tag{6}$$

and such that the fundamental matrix $\Phi(t,s)$ for the function $\mathcal{A}_t := A_t - B_t R_t^{-1} B_t' \Pi_t$ admits the exponential estimate

$$\|\Phi(t,s)\| \leqslant \kappa_1 e^{-\kappa_2 (t-s)}, \quad s \leqslant t, \tag{7}$$

under certain positive constants κ_1, $\kappa_2 > 0$.

We recall that the fundamental matrix $\Phi(t,s)$ is the solution of the problem

$$\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s), \quad \Phi(s,s) = I.$$

If estimate (7) is valid for $\Phi(t,s)$ corresponding to $\mathcal{A}_t$, then $\mathcal{A}_t$ is called the *exponentially stable* matrix (see [13, Definition 2.1, p. 7]), and the action $U_t = K_t X_t$, where K_t is a bounded matrix, leading to such $\mathcal{A}_t$ in (1), is called the *exponentially stable law.* There are various sufficient conditions related with the properties of the linear control systems which ensure satisfaction of Assumption $\mathcal{P}$. For example, such requirements include stabilizability and detectability (see [13, Theorem 2.2, p. 21; 14]).

If Assumption $\mathcal{P}$ is valid, then there exists a stable control law U^* given by the linear feedback

$$U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*, \tag{8}$$

where the process X_t^*, $t \geqslant 0$, is given by

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dw_t, \quad X_0^* = x. \tag{9}$$

As was noticed earlier (see also [15]), the form of U^* can be obtained by the passage to the limit for $T \to \infty$ in the control law $U_t^{*T} = -R_t^{-1} B_t' \Pi_t^T X_t^{*T}$, where $\Pi_t^T \geqslant 0$ is the solution of (6) with the boundary condition $\Pi_T^T = 0$, the process X_t^{*T}, $t \leqslant T$, being determined from (1) for $U_t = U_t^{*T}$. At that, it is known (see [16, Theorem 3.9, p. 301]) that $\mathrm{E} J_T(U^{*T}) = \inf\limits_{U \in \mathcal{U}} \mathrm{E} J_T(U)$, that is, there exists a relation between the stable law and the solution of the control problem over finite intervals.

When considering the long-run averages, we rely on the approach proposed in [17] where the optimality of the control U^* was associated with the solution of problem $\limsup\limits_{T \to \infty} \mathrm{E}\mathcal{K}_T(U) \to \inf\limits_{U \in \mathcal{U}}$ or $\limsup\limits_{T \to \infty} \mathcal{K}_T(U) \to \inf\limits_{U \in \mathcal{U}}$ a.s. The former case suggests the optimality in the average over an infinite time horizon in the criterion $\mathcal{K}$; the latter case suggests stochastic (pathwise) optimality. The long-run average corresponds to $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E} J_T(U)/T$, the extended long-run average is obtained for $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E} J_T(U) / \int_0^T \|G_t\|^2 \, dt$. Introduce the following definition.

Definition 1. Let U^* be a stable control law optimal in the average over an infinite time horizon with respect to the criterion $\mathcal{K}$ in system (1), (2). The criterion $\mathcal{K}$ is called

(a) efficient if $0 < \limsup_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^*) < \infty$ for $\limsup_{T\to\infty} \mathrm{E}J_T(U^*) > 0$;

(b) inefficient if there exists a set $\mathcal{U}^{\mathcal{E}} \subseteq \mathcal{U}$ such that $\limsup_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^*) = \limsup_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^\epsilon) = 0$ for any $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$.

Some auxiliary results concerning the asymptotic behavior of the functionals like (2) are required for the further study of the efficiency of the criteria for long-run average. The exponentially stable laws will be considered as the control strategies because at that the properties of the corresponding trajectories X_t are similar to X_t^*. Therefore, we consider an n-dimensional stochastic process Z_t satisfying the equation

$$dZ_t = \bar{A}_t Z_t dt + G_t dw_t, \qquad Z_0 = z, \tag{10}$$

where the bounded matrix $\bar{A}_t$ is exponentially stable and z is a non-random vector of the initial state.

Lemma 1. *Let M_t be an $n \times n$ bounded matrix. Then, for the process $Z_t,\ t \geqslant 0$, representing solution* (10) *there exists a constant $c_z > 0$, such that*

$$\limsup_{T\to\infty} \frac{\left|\int_0^T E(Z_t' M_t Z_t)dt\right|}{\int_0^T \|G_t\|^2 dt} < c_z.$$

Lemma 1, as well as all subsequent lemmas and theorems, are proved in the Appendix.

Lemma 2. *Let M_t be an $n \times n$ bounded matrix. Then, for the process $Z_t,\ t \geqslant 0$, defined by* (10), *Valid is the relation*

$$\lim_{T\to\infty} \frac{\int_0^T [Z_t' M_t Z_t - E(Z_t' M_t Z_t)]\, dt}{\int_0^T \|G_t\|^2 dt} = \xi \quad a.s.,$$

where

(a) $\xi = 0$ *if* $\int_0^T \|G_t\|^2\, dt \to \infty$, $T \to \infty$;

(b) ξ *is a random variable if* $\int_0^\infty \|G_t\|^2\, dt < \infty$.

3. MAIN RESULTS AND AN EXAMPLE OF INVESTIGATING THE CRITERIA

3.1. Main Results

Passing to the analysis of the long-run averages, we turn to Definition 1. Inefficiency of $\mathcal{K}$ implies that at using this criterion the optimal stable control law U^* is "indistinguishable" in terms of the introduced estimate of application performance from the rest of strategies. The efficient criterion is constructed so as to take into consideration as accurately as possible the order of variation of the expected value of the objective functional. By paying attention to (A.4), one can, in particular, specify the main factor of disturbance actions. Of interest is the situation where the variance $\int_0^T \|G_t\|^2\, dt$ of the integral perturbations grows slower than the planning interval T.

Assumption $\mathcal{G}$.

$$\lim_{T\to\infty} \frac{\int_0^T \|G_t\|^2 dt}{T} = 0. \tag{11}$$

For the case of $\mathcal{G}$, we demonstrate by constructing the set $\mathcal{U}^{\mathcal{E}}$ from Definition 1 that the criterion for long-run average is inefficient. It follows from boundedness of the function B_t that there exists a constant $\bar{b} > 0$ such that $\|B_t\| \leqslant \bar{b},\ t \geqslant 0$. Since Assumption $\mathcal{P}$ is satisfied, the set of numbers $\mathcal{E}$ can be defined as follows:

$$\mathcal{E} = \{\epsilon > 0\ :\ \kappa_3 = \kappa_2 - \epsilon\kappa_1\bar{b} > 0\}.$$

For $\epsilon \in \mathcal{E}$, consider the control

$$U_t^\epsilon = (-R_t^{-1}B'\Pi_t + \epsilon I)X_t^\epsilon,$$

where the process X_t^ϵ, $t \geqslant 0$, is defined by the equation

$$dX_t^\epsilon = (A_t - BR_t^{-1}B'\Pi_t + \epsilon B_t)X_t^\epsilon dt + G_t dw_t, \quad X_0^\epsilon = x, \tag{12}$$

and define the set of controls $\mathcal{U}^{\mathcal{E}} = \{U^\epsilon, \epsilon \in \mathcal{E}\}$. We notice that U^* corresponds to the case $\epsilon = 0$ and, consequently, $U^* \in \mathcal{U}^{\mathcal{E}}$. The following fact can be readily verified.

Lemma 3. *Let Assumption $\mathcal{P}$ be satisfied, and $\epsilon \in \mathcal{E}$ be fixed. Then, the matrix*

$$A_t - BR_t^{-1}B_t'\Pi_t + \epsilon B_t$$

in Eq. (12) *is exponentially stable.*

The following result concerns estimation of growth of $\mathrm{E}J_T(U^\epsilon)$ for $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$ showing inefficiency of using the criterion for long-run average.

Theorem 1. *Let Assumption $\mathcal{P}$ be satisfied. Then, for any $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$ there exists a constant $c_\epsilon > 0$ such that*

$$\limsup_{T\to\infty} \frac{EJ_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt} < c_\epsilon. \tag{13}$$

If at that Assumption $\mathcal{G}$ is valid, then

$$\lim_{T\to\infty} \frac{EJ_T(U^\epsilon)}{T} = 0. \tag{14}$$

At studying efficiency of the pathwise criteria, one can use an analog of Definition 1 where $E\mathcal{K}_T$ is replaced by $\mathcal{K}_T$ and $\mathrm{E}J_T$ by J_T, and also assume the following.

Assumption $\mathcal{G}1$.

$$\int_0^T \|G_t\|^2 dt \to \infty, \quad T \to \infty. \tag{15}$$

Now we formulate the result concerning the values of the pathwise criteria on the controls from the set $\mathcal{U}^{\mathcal{E}}$ and also demonstrating inefficiency of the pathwise ergodic.

Theorem 2. *Let Assumptions $\mathcal{P}$ and $\mathcal{G}1$ be satisfied. Then, for any $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$ with the probability one*

$$\limsup_{T\to\infty} \frac{J_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt} = \limsup_{T\to\infty} \frac{EJ_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt}. \tag{16}$$

If at that Assumption $\mathcal{G}$ is valid, then

$$\lim_{T\to\infty} \frac{J_T(U^\epsilon)}{T} = 0 \quad a.s. \tag{17}$$

Interestingly, according to (16) the value of the stochastic (pathwise) extended long-run average from problem (5) is non-random on the set $\mathcal{U}^{\mathcal{E}}$. More exactly, it coincides with the result of applying criterion for mean, which can be considered an a analog of the well-known ergodicity property of the autonomous control systems, see, for example, [1].

Remark 1. In distinction to the long-run averages, as one can judge from representation (A.4), the efficiency of the criteria for extended long-run average is independent of the specificity of the noise parameters. Here, the key role is played by the behavior of the matrix Π_t which is determined on the basis of the parameters of the deterministic control system.

1. As follows from (A.4) and Theorem 2, for the criteria for extended long-run average to be efficient it suffices that the solution Π_t of the Riccati equation (6) be bounded away from zero, that is, $\Pi_t \geqslant \alpha I,\ t \geqslant 0$, under a certain constant $\alpha > 0$. This property of Π_t is known (see [18, Assertion 16, p. 137]) to take place if the matrix pair $(A_t, \sqrt{Q_t})$ is uniformly completely observable under the conditions of Assumption $\mathcal{P}$. In particular, we note that this requirement is satisfied for $Q_t \geqslant qI,\ t \geqslant 0$, where $q > 0$ is a constant.

2. If $\|\Pi_t\| \to 0,\ t \to \infty$, then the criteria (4) and also (5) are inefficient if Assumption $\mathcal{G}1$ is valid. Indeed, consider $\mathcal{U}^{\mathcal{E}}$ with the controls $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$, where ϵ is an arbitrary real number and $U_t^\epsilon = (-R_t^{-1}B'\Pi_t + \epsilon K_t)X_t^\epsilon$; at that, the matrix K_t is such that $\|K_t\| \to 0,\ t \to \infty$. The process X_t^ϵ, $t \geqslant 0$, obeys equation $dX_t^\epsilon = (A_t - BR_t^{-1}B'\Pi_t + \epsilon B_t K_t)X_t^\epsilon dt + G_t dw_t$, $X_0^\epsilon = x$, its matrix is also exponentially stable (see, for example, [19, Theorem 4.4.6, p. 127]). Therefore, one can take advantage of (A.3), (A.4) and Lemma 2 to obtain the relation

$$\limsup_{T\to\infty} \left(\mathrm{E} J_T(U^\epsilon) \Big/ \int_0^T \|G_t\|^2\, dt \right) = \limsup_{T\to\infty} \left(J_T(U^\epsilon) \Big/ \int_0^T \|G_t\|^2\, dt \right) = 0 \quad \text{a.s.}$$

3.2. Example of Studying the Criteria

We notice that the common long-run average is less risk-sensitive than the extended long-run average relative to the variation of the criterion values. Indeed, the quadratic objective risk functional can also be interpreted as the risk functional of deviation of the trajectory and control [20]. On the other hand, normalization of T corresponds to the case of constant perturbation variance (risk due to the action of uncertainty on the system), and the actual dynamics of the diffusion matrix exerts no influence on the relation between the risk functional and the measure of variation of the integral noise impact. Estimation of the performance of the stable control laws in the one-dimensional problems with identical (constant) values of the parameters $A_t = a$, $B_t = b$, and $R_t = 1$, but different G_t can be considered as an example. For existence of U^*, it suffices to assume that, first, $q > 0$ which corresponds to observability of the deterministic control system, as well as $b \neq 0$ for controllability (see [16, Section 3.4.2, p. 267]). Since the parameters are constant, instead of the solution of the differential Riccati equation one can consider a positive root of the algebraic Riccati equation given by

$$2a\Pi - b^2\Pi^2 + q = 0$$

and equal to

$$\Pi = \frac{a + \sqrt{a^2 + b^2 q}}{b^2}.$$

The above conditions for the parameters also ensure the exponential estimate (7) of the function $\Phi(t,s) = \exp\{-(t-s)\sqrt{a^2+b^2}\}$. Consequently, Assumption $\mathcal{P}$ is satisfied and the stable control law is defined:

$$U_t^{*(i)} = -b^2\Pi X_t^{*(i)},$$

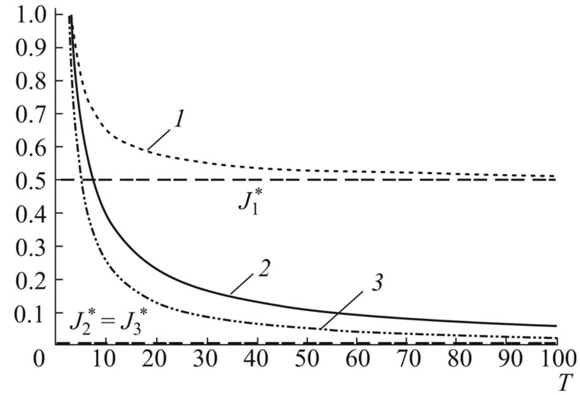

Fig. 1. Study of the long-run average: (*1*) l_T^1, (*2*) l_T^2, (*3*) l_T^3.

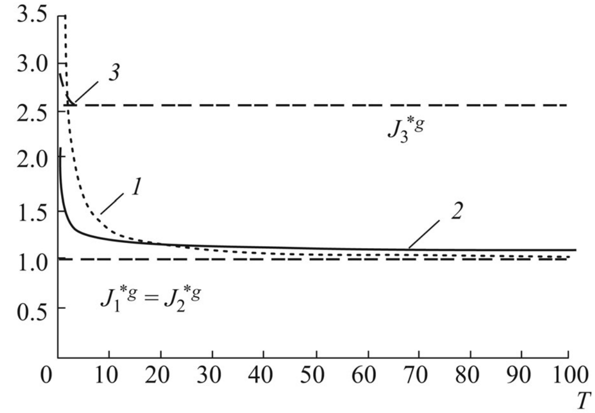

Fig. 2. Study of the extended long-run average: (*1*) g_T^1, (*2*) g_T^2, (*3*) g_T^3.

where $X_t^{*(i)}$ $(i = 1, 2, 3)$ is the notation of the process obtained from the general dynamics Eq. (9) under a corresponding $G_t^{(i)}$:

$$dX_t^{*(i)} = -b^2\Pi X_t^{*(i)}dt + G_t^{(i)}dw_t, \quad X_0^{*(i)} = x,$$

where $G_t^{(1)} = \sin t$, $G_t^{(2)} = 0.55/\sqrt{t+1}$, $G_t^{(3)} = e^{-t/2}$.

The lower estimate of the criterion $J_i^{*g} = \limsup\limits_{T\to\infty}(\mathrm{E}J_T(U^{(i)})/\int_0^T(G_t^{(i)})^2\,dt)$ for the extended long-run average $(i = 1, 2, 3)$ in all three cases equals Π (see also the representation (A.4)):

$$\Pi = J_1^{*g} = J_2^{*g} \leqslant \Pi(x^2+1) = J_3^{*g}.$$

In the case of $G_t^{(3)} = e^{-t/2}$, the value of the criterion includes also an additional term Πx^2 showing the contribution of the initial state in the case of fastly damping perturbations. Here we see also an analog to the deterministic linear regulator.

If we consider the long-run average, then only in the first case the value is equal to a constant smaller than Π:

$$J_1^* = \frac{\Pi}{2} = \limsup_{T\to\infty}\frac{\mathrm{E}J_T(U^{*(1)})}{T},$$

in the two remaining situations (this result was established within the framework of Theorem 1) the result is equal to zero:

$$J_2^* = J_3^* = \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^{*(2)})}{T} = \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^{*(3)})}{T} = 0.$$

At that, $\mathrm{E}J_T(U^{*(2)})/T$ decreases asymptotically as $\ln T/T$, and for $\mathrm{E}J_T(U^{*(2)})/T$ in turns out that the order of variation is about $1/T$. In the above example, the long-run average tends to lower values, which takes place even in the case where the variance of the integral noise actions grows in proportion to T. The graphs depict $l_T^i = \sup_{t\geqslant T}\{\mathrm{E}J_t(U^{*(i)})/t\}$ and $g_T^i = \sup_{t\geqslant T}\left\{\mathrm{E}J_t(U^{*(i)})/\int_0^T (G_s^{(i)})^2\,ds\right\}$ vs. the planning horizon T. They underlie calculation of the criteria of long-run (Fig. 1) and extended long-run (Fig. 2) averages, as well as the values of J_i^* and J_i^{*g} themselves, $i = 1, 2, 3$. The calculations were carried out for the parameters $a = 0$, $b = q = 1$, and $x = 1.25$; at that, $\Pi = 1$, $T \in (0, 100]$. We also notice that for the extended long-run average the difference between g_T^i and J_i^{*g} has the order of decrease equal to $1/\int_0^T (G_t^{(i)})^2\,dt$.

Therefore, in the long-run average no difference is made between the impact of noise on the system dynamics within the long-run period. On the contrary, the criterion for the extended long-run average takes into account the balance in the relation between the quadratic risk functional and risk caused by a random impact on the system.

4. EXTENDED LONG-RUN AVERAGE AND THE DISCOUNTED SYSTEMS

Consider a discounted control system with state $\tilde{X}_t,\ t \geqslant 0$, defined by an n-dimensional controllable stochastic process with the dynamics given by

$$d\tilde{X}_t = A\tilde{X}_t dt + B\tilde{U}_t dt + G dw_t, \qquad \tilde{X}_0 = x, \tag{18}$$

where A, B, and G, $\|G\| > 0$, are constant matrices, the admissible control $\tilde{U}_t$ is defined as in (1), and $\mathcal{U}$ is the set of admissible controls.

The objective functional which includes the discount function f_t is given by

$$\tilde{J}_T(\tilde{U}) = \int_0^T f_t(\tilde{X}_t' Q \tilde{X}_t + \tilde{U}_t' R \tilde{U}_t)\,dt, \tag{19}$$

where $\tilde{U}_t$ are the admissible controls $\tilde{U} \in \mathcal{U}$ over the interval $[0, T]$, $Q \geqslant 0$ and $R > 0$ are constant symmetrical matrices, $f_t > 0$, $f_0 = 1$ is the discount function assumed to be nonincreasing and differentiable, $f_t \to 0$, $t \to \infty$, and the discount rate $\phi_t = -\dot{f}_t/f_t$, $t \geqslant 0$, is bounded. The discount functions satisfying the above assumptions are exemplified by the traditional exponential function with $f_t = e^{-\gamma t}$ $(\gamma > 0)$; hyperbolic discount function for $f_t = 1/(1+\theta t)^{\theta_1/\theta}$ $(\theta_1, \theta > 0)$; and the so-called double discounting $f_t = m_1 e^{-\alpha t} + (1 - m_1)e^{-\beta t}$ $(\alpha, \beta > 0,\ 0 < m_1 < 1)$.

Obviously, (18), (19) do not belong to the aforementioned class of systems (1), (2) in view of unboundedness of the matrix $f_t R$ away from zero. Indeed, in this case the Riccati equation

$$\dot{\tilde{\Pi}}_t + \tilde{\Pi}_t A + A'\tilde{\Pi}_t - (1/f_t)\tilde{\Pi}_t B R^{-1} B' \tilde{\Pi}_t + f_t Q = 0, \tag{20}$$

companion of (6), includes unbounded coefficients and gives rise to the question of existence of the stable control law

$$\tilde{U}_t^* = -R^{-1}B'(\tilde{\Pi}_t/f_t)\tilde{X}_t^*. \tag{21}$$

Additionally, it is required to construct an adequate criterion for optimality over an infinite time horizon. As will be clear from the following reasoning, the extended long-run average is inefficient for the discounted systems. The questions of determining the optimal control and selection of the criterion for (18), (19) were discussed in [2, 3, 11]. System (18), (19) was shown to be reducible by the change of variables $X_t = \sqrt{f_t}\tilde{X}_t$, $U_t = \sqrt{f_t}\tilde{U}_t$ to (1), (2) with $A_t = A - (1/2)\phi_t I$, $B_t = B$, $G_t = \sqrt{f_t}G$, $Q_t = Q$, and $R_t = R$. The values of the objective functionals coincide, that is, $J_T(U) = \tilde{J}_T(\tilde{U})$, and the criterion for extended long-run average (4) may be used in the new system $\limsup\limits_{T\to\infty}(\mathrm{E}J_T(U)/\int_0^T f_t\|G\|^2\,dt)$. If for the aforementioned matrices of parameters Assumption $\mathcal{P}$ is valid, then there exists a stable optimal control law (8) like $U_t^* = -R^{-1}B'\Pi_t X_t^*$, where X_t^* is defined by (9) and the bounded matrix function Π_t satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t(A - (1/2)\phi_t I) + (A' - (1/2)\phi_t I)\Pi_t - \Pi_t B R^{-1} B'\Pi_t + Q = 0. \tag{22}$$

By returning to the original discounted system, we have

$$\begin{gathered}\tilde{U}_t^{*d} = -R^{-1}B'\Pi_t\tilde{X}_t^{*d},\\ d\tilde{X}_t^{*d} = (A - BR^{-1}B'\Pi_t)\tilde{X}_t^{*d}dt + G dw_t, \quad \tilde{X}_0^{*d} = x.\end{gathered} \tag{23}$$

It is also known that control (23) is optimal (see [11]) with respect to the criterion of expected long-run loss per unit of the cumulated discount

$$\limsup_{T\to\infty}\frac{\mathrm{E}\tilde{J}_T(\tilde{U})}{\int\limits_0^T f_t\,dt} \to \inf_{\tilde{U}\in\mathcal{U}}. \tag{24}$$

Passing to determination of $\tilde{U}^*$ from (21), compare (20) and (22) from which one can readily see that $\Pi_t = \tilde{\Pi}_t/f_t$. Therefore, existence of Π_t ensures existence of the solution also for Eq. (20). Moreover, at that $\tilde{U}_t^* = \tilde{U}_t^{*d}$, $\tilde{X}_t^* = \tilde{X}_t^{*d}$. It follows from the properties of the discount function f_t and the above relation between Π_t and $\tilde{\Pi}_t$ that $\|\tilde{\Pi}_t\| \to 0$, $t\to\infty$. Then, by item 2 of Remark 1 for the extended long-run average for the discounted system the criterion is inefficient. Use of the expected long-run loss per unit of the cumulated discount (24) is substantiated by the above reasoning about rearrangement of (18), (19) to the standard control system (1), (2) and its subsequent optimization.

5. CONCLUSIONS

The present paper analyzed the optimality criteria used in the problem of control of the linear stochastic systems over an infinite time horizon. The long-run average, as well as the pathwise ergodic average, are inefficient because they can take zero values for an entire set of controls other than the optimal one. This fact was established by determining the order of growth of the objective functionals on the exponentially stable control laws. The key part is played here by the variability of the parameters of the perturbing process, and this distinction is taken into consideration in the structure of the new criterion called the extended long-run average, as well in its stochastic counterpart. To be more specific, normalization is used in the form of a sum of variances of the components of the vector of integral noise actions, which allows one to get a criterion efficient in terms of the factor under consideration. At the same time, for the discounted control systems the solution of the Riccati equation defining the stable optimal control law tends to zero. As was noted above, this leads to inefficiency of the criteria for long-run average. In such situation, it is worthwhile to use the expected long-run loss per unit of the cumulated discount, which is justified by the relationship of the discounted control system and the standard problem including the criterion for extended long-run average. Construction and efficiency analysis of the optimality criteria in the tracking problems for the stochastic linear regulator and also consideration of the case of growing perturbations ($\|G_t\| \to \infty$, $t \to \infty$, see [21]) can be mentioned as the lines for future research.

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APPENDIX

Proof of Lemma 1. Consider the linear matrix differential equation

$$\dot{P}_t + \bar{A}_t' P_t + P_t \bar{A}_t + M_t = 0, \tag{A.1}$$

with solution representable as $P_t = \int_t^\infty \Phi'(s,t) M_s \Phi(s,t)\, ds$ and bounded in virtue of the matrices $\bar{A}_t$ and M_t.

Using (10) and (A.1), by the Ito isometry we get $d(Z_t' P_t Z_t) = -Z_t' M_t Z_t dt + \mathrm{tr}(G_t' P_t G_t) dt + 2Z_t' P_t G_t dw_t$. Therefore, we get the representation

$$\int_0^T Z_t' M_t Z_t\, dt = z P_0 z - Z_T' P_T Z_T + \int_0^T \mathrm{tr}(G_t' P_t G_t)\, dt + 2 \int_0^T Z_t' P_t G_t\, dw_t, \tag{A.2}$$

and for the expected value

$$\int_0^T \mathrm{E}(Z_t' M_t Z_t)\, dt = z' P_0 z - \mathrm{E}(Z_T' P_T Z_T) + \int_0^T \mathrm{tr}(G_t' P_t G_t)\, dt. \tag{A.3}$$

We note that for $\bar{A}_t = A_t - B_t R_t^{-1} B_t' \Pi_t$, $z = x$, the process $Z_t = X_t^*$, $P_t = \Pi_t$ and (A.3) provide

$$\mathrm{E} J_T(U^*) = x' \Pi_0 x - \mathrm{E}[(X_T^*)' \Pi_T X_T^*] + \int_0^T \mathrm{tr}(G_t' \Pi_t G_t)\, dt. \tag{A.4}$$

To estimate the second term in (A.3), we write down the solution of Eq. (10):

$$Z_t = \Phi(t,0) z + \int_0^t \Phi(t,s) G_s dw_s,$$

where $\Phi(t,s)$ is the fundamental matrix for the function $\bar{A}_t$ admitting an exponential estimate like (7) with the constants $\bar{\kappa}_1, \bar{\kappa}_2 > 0$. Then, for $c > 0$ in virtue of the Ito isometry

$$\mathrm{E}\|Z_t\|^2 \leqslant c \left(e^{-2\bar{\kappa}_2 t} \|z\|^2 + \int_0^t e^{-2\bar{\kappa}_2 (t-s)} \|G_s\|^2 ds \right).$$

Here and below c denotes a positive constant whose particular value is of no importance and can vary from formula to formula. At that, it is evident that $\mathrm{E}\|Z_t\|^2$ is bounded. Returning to (A.3), we have $\left| \int_0^T \mathrm{E}(Z_t' M_t Z_t)\, dt \right| \leqslant c(\|z\|^2 + \int_0^T \|G_t\|^2\, dt)$, which proves Lemma 1.

Proof of Lemma 2. To find the representation

$$\frac{\int_0^T [Z_t' M_t Z_t - \mathrm{E}(Z_t' M_t Z_t)]\,dt}{\int_0^T \|G_t\|^2 dt} = \frac{\mathrm{E}(Z_T' P_T Z_T) - Z_T' P_T Z_T}{\int_0^T \|G_t\|^2\,dt} + 2\frac{\int_0^T Z_t' P_t G_t\,dw_t}{\int_0^T \|G_t\|^2 dt}, \tag{A.5}$$

we use (A.2) and (A.3) and notice that in virtue of boundedness of P_t and exponential stability of $\bar{A}_t$ we can apply the result of [3, Theorem 1] on asymptotic behavior of the normalized process $\|Z_T\|^2/\int_0^T \|G_t\|^2 dt$ to the first term in (A.5), whence follows the tendency of the expression under consideration a.s. to zero for $T\to\infty$ (here we used the relations $\mathrm{E}\|Z_T\|^2 \leqslant c$ and $\mathrm{E}\|Z_T\|^2 \to 0$ if $\int_0^\infty \|G_t\|^2 dt < \infty$). It was also shown in [3] that in the case of $\int_0^T \|G_t\|^2 dt \to \infty$, $T\to\infty$, the condition $\mathrm{E}\|Z_t\|^2 \leqslant c$ provides convergence of $(\int_0^T Z_t' P_t G_t dw_t)/\int_0^T \|G_t\|^2 dt \to 0$ with the probability one (see the proof of Theorem 2 in [3]). If $\int_0^\infty \|G_t\|^2 dt < \infty$, then there exists the random variable $\xi = (\int_0^\infty Z_t' P_t G_t dw_t)/\int_0^\infty \|G_t\|^2 dt$ (see [3, Lemma 3]), which proves Lemma 2.

Proof of Lemma 3. Consider the matrix $\mathcal{A}_t := \tilde{A}_t + \tilde{B}_t$, where $\tilde{A}_t$ is exponentially stable and $\|\tilde{B}_t\| \leqslant \tilde{b}$ $\tilde{b} > 0$ being of constant. Then, by an assertion in [19, Theorem 4.4.6, p. 127] the fundamental matrix $\Phi(t,s)$ corresponding to $\mathcal{A}_t = \tilde{A}_t + \tilde{B}_t$ admits the estimate $\|\Phi(t,s)\| \leqslant \kappa_1 e^{-\kappa_3(t-s)}$, $s \leqslant t$, where $\kappa_3 = \kappa_2 - \tilde{b}$. Assuming that $\tilde{A}_t = A_t - B_t R_t^{-1} B_t' \Pi_t$, $\tilde{B}_t = \epsilon B_t$, from the condition $\kappa_3 = \kappa_2 - \epsilon\bar{b} > 0$ we establish the desired property of the matrix $A_t - BR_t^{-1}B_t'\Pi_t + \epsilon B_t$, which proves Lemma 3.

Proof of Theorem 1. By Lemma 3, the matrix $A_t - BR_t^{-1}B_t'\Pi_t + \epsilon B_t$ in Eq. (12) is exponentially stable. Relation (13) is obtained by applying Lemma 1 under $Z_t = X_t^\epsilon$ and $M_t = Q_t + L_t' R_t L_t$, where $L_t = -R_t^{-1}B_t'\Pi_t + \epsilon B_t$. Satisfaction of Assumption $\mathcal{G}$ leads to (14), which proves Theorem 1.

Proof of Theorem 2. The result (16) follows from Lemma 2 if we assume that $Z_t = X_t^\epsilon$, $M_t = Q_t + L_t' R_t L_t$, where $L_t = -R_t^{-1}B_t'\Pi_t + \epsilon B_t$ and set down the representation

$$\frac{J_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt} = \frac{J_T(U^\epsilon) - \mathrm{E}J_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt} + \frac{\mathrm{E}J_T(U^\epsilon)}{\int_0^T \|G_t\|^2\,dt}.$$

At that, (17), obviously, follows from (14), which proves Theorem 2.

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