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Stabilization of Linear Stochastic Systems with a Discount: Modeling and Estimation of the Long-Term Effects from the Application of Optimal Control Strategies

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Abstract—The paper is devoted to the problem of stabilizing a linear stochastic control system. The quadratic cost functional measures the total loss caused by deviation from the fixed (target) levels and control trajectories, as well as a decision-maker's time preferences expressed in the discount function. The long-term impacts of the use of decision-making, optimal on average, over an infinite-time horizon are taken as estimates of the deviation of the optimal trajectory from its target in the mean square sense and with the probability of 1.

Keywords: stabilization of linear systems, quadratic functional, time preferences, discounting function, infinite-time horizon of planning

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1. INTRODUCTION

Linear stochastic control systems are widely used in modeling the processes in different fields of economics: macroeconomic regulation, economics of transport, ecological economics, insurance, etc. In the choice of control actions within long planning intervals, one of the important tasks is the stabilization of the system, i.e., support of its evolution trajectory close to the earlier chosen state. For example, for macro economic stabilization, the variables to be corrected can be inflation, unemployment, etc. [1, 2]; in ecological economics, it can be the regulation of the volume of harmful waste [3]. Here it is meant that any deviation of the current state of the process and control from the target leads to losses, and in their estimation, account is taken for the time preferences of the agents realizing these strategies. Time preferences are at the core of the differences in their estimation of future events ("gain" or "loss") and influence their decisions, thus making them an important socioeconomic aspect of modeling. "Positive" time preferences mean the desire to speed up "gains" and to postpone "losses". The "negative" time preferences denote the reverse situation, and in zero preferences the factor of time has no role in the choice. It is assumed that time preferences can be expressed in terms of the discounting function f_t, determining the "weight" of the event to occur at the time moment $t \geq 0$ at its estimation at the initial moment. The discounting function decreases for positive time preferences, increases for negative time preferences, and remains constant at the zero time preferences.

In the mathematical modeling of the stabilization process, a number of problems arise. The first is to assess the quality of the chosen strategy of control, i.e., to determine the utility function for the planning period and to form a criterion of optimality under a planning horizon tending to infinity. It is natural to assign the optimality functional in the integral quadratic form [4] and to include the discounting function there reflecting the subjects' time preferences. For the decisions to be made on unbounded planning horizons, a criterion is used that includes the minimization of the upper limit from the expected losses per unit of the accumulated discount [3, 5]. The thus defined law of control will be called the average optimal over an infinite-time horizon. Another important problem is the evaluation of the long-term effects from the application of the optimal strategy, i.e., how far the corresponding trajectory of the system's growth will correspond to the planned level. Here it is supposed to consider two kinds of assessments of the deviations of the controlled process from the planned value—in the mean square and with probability 1.

It is also of interest to find how far the choice of the optimal control in the use of the functional with discounting promotes the achievement of the target in the stabilization of the trajectory, taking into account the differences in the priority of the future for the subjects. Note that in the analysis of the macroeconomic model formulated in [6] as a problem of a linear quadratic regulator, a proposition was made on the stabilizing effect of the negative time preferences but on the destablilizing effect of the positive ones in the use of the exponential discounting function; see also [7]. Now the work is organized in the following way. Section 2 is devoted to the description of the model and to finding the optimal control law. Section 3 considers the long-term consequences of the use of this strategy. The basic conclusions are formulated in Section 4.

2. DESCRIPTION OF THE MODEL AND DETERMINATION OF THE OPTIMAL CONTROL

Consider the linear economic control system under uncertainty. Let on the complete probability space $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ an n-dimensional stochastic process X_t, $t \geq 0$, determining the state of the system and described by equation

$$dX_t = AX_t dt + BU_t dt + Gdw_t, \quad X_0 = x, \tag{1}$$

be given, where the initial state x is a non-random; w_t, $t \geq 0$, is a d-dimensional standard Wiener process; U_t, $t \geq 0$, is an admissible control, or the k-dimensional stochastic process, adapted to filtration $\{\mathfrak{F}_t\}_{t\geq 0}$, and $\mathfrak{F}_t = \sigma\{w_s, s \leq t\}$, such that Eq. (1) has the solution; A, B, and G are matrices of such dimensionalities, under which (1) makes sense; for matrix G parameters of the perturbing process it is supposed that $\|G\| > 0$ ($\|\cdot\|$ is the matrix's Eucleadean norm). The set of admissible controls will be denoted as $\mathcal{U}$.

Now let us go to the mathematical description of the problem of stabilization. Consider the determined process described by Eq. (1) at $G = 0$ (0 is the zero matrix), and let x_0 be the fixed vector. Suppose there is control $U_t \equiv u_0$, at which $X_t \equiv X_0 = x_0$ is the stationary state of this process, i.e., $Ax_0 + Bu_0 = 0$ (0 is the zero vector). Note that, for example, at $n = 1$ and $B \neq 0$ such control always exists.

We define the cost functional (a random variable), which takes account for the losses caused by deviation of process X_t and admissible control U_t from x_0 and u_0, respectively, as well as the dependent subjective estimate of these losses (i.e., time preferences) for the planning period $[0, T]$:

$$J_T(U) = \int_0^T f_t \cdot [(X_t' - x_0')Q(X_t - x_0) + (U_t' - u_0')R(U_t - u_0)]dt, \tag{2}$$

where $U \in \mathcal{U}$ is admissible control, Q, R are symmetric matrices, positive semidefinite and positive definite, respectively, $'$ is the transpose sign, and f_t is the discounting function reflecting the time preferences of the agents and having properties given in the following assumption.

Assumption $\mathcal{D}$. The discounting function is $f_t > 0$, $t \geq 0$, $f_0 = 1$:

(1) it is monotone and differentiable; if f_t is increasing, then $f_t \to \infty$, $t \to \infty$, if f_t is decreasing, then $f_t \to 0$, $t \to \infty$;

(2) the discounting rate $\phi_t = -\dot{f}_t / f_t$ is a bounded function at any $t \geq 0$ ("$\cdot$" is a time derivative) and $\lim_{t\to\infty} \phi_t = c_\phi$, where c_ϕ is a constant.

We note that a decreasing f_t corresponds to positive time preferences ($\phi_t > 0$), in the case of negative time preferences f_t increases ($\phi_t < 0$), if $f_t \equiv 1$, then the time preferences are zero ($\phi_t \equiv 0$).

To estimate the performance of control actions chosen in order to stabilize the system in a long-term period, consider the criterion of optimality proposed in [3, 5]. We will try to find control $U^* \in \mathcal{U}$ such that the correlation

$$\limsup_{T\to\infty} \frac{\dfrac{EJ_T(U^*)}{T}}{\int_0^T f_t dt} \leq \limsup_{T\to\infty} \frac{\dfrac{EJ_T(U)}{T}}{\int_0^T f_t dt} + c_J \tag{3}$$

is fulfilled for any $U \in \mathcal{U}$ at a constant $c_J \geq 0$, independent of control. In (3) a comparison is made of the upper limits of the relationships, which are the expected losses per unit of the accumulated discount with the use of different control.

Definition 1. *Control $U^* \in \mathcal{U}$ will be called average optimal over an infinite-time horizon if (3) is fulfilled for any $U \in \mathcal{U}$ at $c_J = 0$, and weakly average optimal if $c_J > 0$.*

Suppose control U_t^* has been found that is average optimal (weakly optimal) over an infinite-time horizon, and X_t^* is the corresponding process defined by (1). Here it is investigated how far the application of the average optimal control U^* provides the achievement of the goal to keep X_t^* near x_0 in the long run. The process $D_t = X_t^* - x_0$ is considered, assigning the deviation of the random trajectory X_t^* from the target vector x_0, and two kinds of estimates will be found:

—estimates for the difference in the mean square $E\|D_t\|^2 = E\left\|X_t^* - x_0\right\|^2$;

—estimates in the sense of close to probable (c.t.p.) for $\|D_t\|^2 = \left\|X_t^* - x_0\right\|^2$ in accordance with the definition of the upper function for a stochastic process.

Definition 2. *Function $h_t > 0$ is called the upper function for the process $\|D_t\|^2$, if there is an a.s. finite moment of time t_0 such that $\|D_t\|^2 \le h_t$ holds almost surely for any $t > t_0$.*

Now we provide a number of basic assumptions related to the parameters of the control system, which, as will be given below, guarantee for the existence of control U^*. For systems with discounting these requirements have been formulated in [3].

Assumption $\mathcal{P}$

1. Matrices $A_t = A - (1/2)\phi_t \cdot I$, B, Q, R, $t \ge 0$ such that there is a bounded absolutely continuous function $\Pi_t, t \ge 0$. This function takes values in a set of symmetrical positive semidefinite matrices and satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t'\Pi_t - \Pi_t BR^{-1}B'\Pi_t = 0. \tag{4}$$

In this case the fundamental matrix $\Phi_{\mathcal{A}}(t,s)$ admits for function $\mathcal{A}_t := A_t - BR^{-1}B'\Pi_t$ the exponential estimate

$$\|\Phi_{\mathcal{A}}(t,s)\| \le \kappa_1 \exp\{-\kappa_2(t-s)\}, \quad s \le t, \tag{5}$$

with some positive constants $\kappa_1, \kappa_2 > 0$; I is an identity matrix.

2. There is a constant $c_0 > 0$ such that for any pair $(x_t, u_t)_{t \le T}$, satisfying the equation

$$dx_t = A_t x_t dt + Bu_t dt, \quad x_0 = 0,$$

the inequality

$$\|x_T\|^2 + \int_0^T \|x_t\|^2\, dt \le c_0 \int_0^T (x_t'Qx_t + u_t'Ru_t)dt.$$

is true.

The fundamental matrix $\Phi_{\mathcal{A}}(t,s)$ for function $\mathcal{A}_t$ is defined as the solution of problem

$$\frac{\partial \Phi_{\mathcal{A}}(t,s)}{\partial t} = \mathcal{A}_t \Phi_{\mathcal{A}}(t,s), \quad \Phi(s,s) = I.$$

Definition 3. If $\Phi_{\mathcal{A}}(t,s)$ *admits the estimate* (5)*, then matrix $\mathcal{A}_t$ is called exponentially stable.*

The conditions in assumption $\mathcal{P}$ are common enough and there are various well-known requirements for the properties of the linear control systems, whose fulfillment implies that assumption $\mathcal{P}$ is valid. We should note here also such characteristics as uniform complete controllability of pair (A_t, B) and the uniform complete observability of (A_t, Q) (see [8], [9]).

The statement below provides a optimal control law.

Theorem 1. *Let assumptions $\mathcal{D}$ and $\mathcal{P}$ hold. Then control of the form*

$$U_t^* = -R^{-1}B'\Pi_t(X_t^* - x_0) + u_0, \tag{6}$$

where function Π_t satisfies (4) and *the process $\{X_t^*\}_{t=0}^{\infty}$ satisfies the equation.*

$$dX_t^* = (A - BR^{-1}B'\Pi_t)(X_t^* - x_0)dt + Gdw_t, \quad X_0^* = x, \tag{7}$$

is

(a) *average optimal over an infinite-time horizon, if in assumption $\mathcal{D}$ $c_\phi \ge 0$;*

(b) *weakly average optimal over an infinite-time horizon if in assumption $\mathcal{D}$ $c_\phi < 0$.*

Proof. For bounded f_t (i.e., in positive and zero discounting), the statement of Theorem 1 is a consequence of the application of the result on the average optimality for systems with a time-varying of parameters of the perturbation process, see [5]. Indeed, having changed variables $\tilde{X}_t = \sqrt{f_t}(X_t - x_0)$, $\tilde{U}_t = \sqrt{f_t}(U_t - u_0)$, we obtain the equation of the dynamics of the process $d\tilde{X}_t = (A - (1/2)\phi_t \cdot I)\tilde{X}_t dt + B\tilde{U}_t dt + \sqrt{f_t} G dw_t$, $\tilde{X}_0 = x - x_0$. The objective functional in new variables $\tilde{J}_T(\tilde{U}) = \int_0^T (\tilde{X}_t' Q \tilde{X}_t + \tilde{U}_t' R \tilde{U}_t) dt$, in this $J_T(U) = \tilde{J}_T(\tilde{U})$. Then, on condition of the fulfillment of assumptions $\mathcal{D}$ and $\mathcal{P}$, according to the statement of Theorem 1 from [5], control $\tilde{U}_t^* = -R^{-1}B'\Pi_t \tilde{X}_t^*$ is average optimal over an infinite-time horizon in the sense of the solution of problem $\limsup_{T\to\infty}(E\tilde{J}_T(\tilde{U}) \Big/ \int_0^T f_t \|G\|^2 dt) \to \inf$. The inverse change of variables to those made earlier, as well as assumption $Ax_0 + Bu_0 = 0$, lead to the optimal control U^* of type (6) and to its corresponding process (7) in the initial system with discounting. For the case of the negative discounting (increasing f_t), Theorem 1 was proved in [3, Theorem 3] with assumptions $\mathcal{D}$ and $\mathcal{P}$.

Note 1. If the there is no uncertainty in the system, i.e., $\|G\| = 0$, then, as was shown in [3], instead of problem (3), the standard linear quadratic control problem over an infinite-time horizon can be considered

$$\limsup_{T\to\infty} J_T(U) \to \inf_{U\in\mathcal{U}},$$

and the control law U^* given in Theorem 1 defined by (6)–(7) at $G = 0$ will be the solution of this problem.

In considering difference $D_t = X_t^* - x_0$, based on (7) we write out the equation of the dynamics for this process as

$$dD_t = A_t^* D_t dt + G dw_t, \quad D_0 = x - x_0, \tag{8}$$

where $A_t^* = A - BR^{-1}B'\Pi_t$ is a bounded matrix.

It is well known (see [8, 10]) that the asymptotic behavior of the solutions of linear stochastic differential equation of type (8) depends on the characteristics of the fundamental matrix $\Phi_{A^*}(t,s)$, constructed for function A_t^*. In the following lemma, the statement about the dependence of the property of the exponential stability of matrix A_t^* on the asymptotic discounting rate c_ϕ is formulated.

Lemma 1. *Let assumptions* $\mathcal{D}$ *and* $\mathcal{P}$ *be fulfilled and* κ_2 *is a constant from* (5). *Then*

(*a*) *if* $c_\phi < 2\kappa_2$, *matrix* A_t^* *is exponentially stable*;

(*b*) *if* $c_\phi \ge 2\kappa_2$, *then for the fundamental matrix* $\Phi_{A^*}(t,s)$ *estimate*

$$\|\Phi_{A^*}(t,s)\| \le \mu_1 \exp\{\mu_2(t-s)\}, \quad s \le t, \text{ is true} \tag{9}$$

at some positive constant $\mu_1 > 0$ constant $\mu_2 > c_\phi/2 - \kappa_2$.

Proof. Matrix A_t^* can be represented as $A_t^* = \mathcal{A}_t + (1/2)\phi_t \cdot I$, then on assumption $\mathcal{P}$, $\mathcal{A}_t = A_t - BR^{-1}B'\Pi_t$ is exponentially stable. Then

$$\Phi_{A^*}(t,s) = \Phi_{\mathcal{A}}(t,s)\exp\left\{(1/2)\int_s^t \phi_v dv\right\},$$

from which, taking into account the definition of the asymptotic discounting rate c_ϕ the estimate

$$\|\Phi_{A^*}(t,s)\| \le \mu_1 \exp\{-[\kappa_2 - (1/2)(c_\phi + \varepsilon)](t-s)\}$$

follows with constant $\mu_1 > 0$ and an arbitrarily small number $\varepsilon > 0$.

Under the conditions of (a), by finding $\kappa = \kappa_2 - (1/2)(c_\phi + \varepsilon) > 0$, we obtain the exponential estimate of type (5). In (b) we assign $\mu_2 = (1/2)(c_\phi + \varepsilon) - \kappa_2 > 0$.

Note that for discounting functions with asymptotically nonpositive rate $c_\phi \le 0$ the condition in (a) of Lemma 1 is always fulfilled. When conventional exponential discounting with $f_t = \exp\{-\gamma t\}$ $(\gamma > 0)$ the possibility of an exponential estimate of the fundamental matrix will be determined by the relationship between rate γ and constant κ_2.

3. ESTIMATE OF LONG-TERM EFFECTS FROM THE APPLICATION OF OPTIMAL MANAGEMENT STRATEGIES

At first we define the bounds for $E\|D_t\|^2 = E\left\|X_t^* - x_0\right\|^2$ as the mean of the squared distance between the process on optimal control and its target level.

Theorem 2. *Let the conditions of Theorem* 1 *hold and* κ_2 *be the constant from* (5). *Then the bounds for the mean square difference are defined as follows:*

(*a*) *if* $c_\phi < 2\kappa_2$, *then* $c_L \le E\left\|X_t^* - x_0\right\|^2 \le c_U$,

(*b*) *if* $c_\phi \ge 2\kappa_2$, *then* $c_L \le E\left\|X_t^* - x_0\right\|^2 \le c_U \cdot \exp\{\kappa t\}$,

where $c_L, c_U > 0$, $\kappa > c_\phi - 2\kappa_2$ are positive constants.

Proof. The lower bound in (a) and (b) will be the consequence of the relation

$$E\|D_t\|^2 \ge \bar{\kappa}_1 \exp\{-\bar{\kappa}_2 t\}\|x - x_0\|^2 + \bar{\kappa}_1 \int_0^t \exp\{-\bar{\kappa}_2(t-s)\}\|G\|^2\, ds, \tag{10}$$

which holds with some constants $\bar{\kappa}_1, \bar{\kappa}_2 > 0$. Now we prove the inequality in (10).

Define matrix $P_t = E(D_t D_t')$. It is known (see, for example, [11, p. 98, Theorem 4.2.4]), that P_t satisfies the linear matrix differential equation

$$\dot{P}_t = A_t^* P_t + P_t (A_t^*)' + GG', \quad P_0 = D_0 D_0', \tag{11}$$

and solution (11) is written as

$$P_t = \Phi_{A^*}(t,0) D_0 D_0' \Phi'_{A^*}(t,0) + \int_0^t \Phi_{A^*}(t,s) GG' \Phi'_{A^*}(t,s) ds.$$

Note also that

$$E\|D_t\|^2 = \mathrm{tr}(P_t) = \mathrm{tr}\{\Phi_{A^*}(t,0) D_0 D_0' \Phi'_{A^*}(t,0)\} + \int_0^t \mathrm{tr}\{\Phi_{A^*}(t,s) G_s G_s' \Phi'_{A^*}(t,s)\} ds,$$

where $\mathrm{tr}(\cdot)$ is the matrix trace.

As function A_t^* is bounded, then (see [12, p. 40, Theorem 5]) there are constants $\bar{\kappa}_1, \bar{\kappa}_2 > 0$ such that

$$\Phi'_{A^*}(t,s)\Phi_{A^*}(t,s) \ge \bar{\kappa}_1 \exp\{-\bar{\kappa}_2 (t-s)\} \cdot I, \quad s \le t,$$

which is equivalent to relationship

$$y' \Phi'_{A^*}(t,s) \Phi_{A^*}(t,s) y \ge \bar{\kappa}_1 \exp\{-\bar{\kappa}_2(t-s)\}\|y\|^2, \quad s \le t, \tag{12}$$

for an arbitrary vector y of the corresponding dimension. Note that

$$\mathrm{tr}\{\Phi_{A^*}(t,s) GG' \Phi'_{A^*}(t,s)\} = \sum_{j=1}^{d} (G^j)' \Phi'_{A^*}(t,s) \Phi_{A^*}(t,s) G^j, \tag{13}$$

where G^j is the jth column of matrix G.

Putting sequentially that $y = G^j$ in (12) and taking into account (13), we obtain the estimate

$$\mathrm{tr}\{\Phi_{A^*}(t,s) GG' \Phi'_{A^*}(t,s)\} \ge \bar{\kappa}_1 \exp\{-\bar{\kappa}_2(t-s)\}\|G\|^2,$$

using which and the relation $\mathrm{tr}\{\Phi_{A^*}(t,0)D_0D_0'\Phi'_{A^*}(t,0)\} \geq \bar{\kappa}_1 \exp\{-\bar{\kappa}_2 t\}\|D_0\|^2$ in integration, we come to (10). Thus, the inequality for the lower bound has been proved.

Moving to the proof of the type of the upper estimate in (a) and (b), from (8) by the Ito isometry we write out the estimate

$$E\|D_t\|^2 \leq \|\Phi_{A^*}(t,0)\|^2\|D_0\|^2 + \int_0^t \|\Phi_{A^*}(t,s)\|^2\|G\|^2 ds. \tag{14}$$

As has been shown (see Lemma 1), from the conditions of (a) of the statement being proved it follows the exponential stability of matrix A_t^*. Using this property, we transform (14) to the form

$$E\|D_t\|^2 \leq \mu_1^2 \exp\{-2\kappa t\}\|D_0\|^2 + \mu_1^2 \int_0^t \exp\{-2\kappa(t-s)\}\|G\|^2 ds \tag{15}$$

with some constants $\mu_1, \kappa > 0$. It is easy to see that the expression in the right-hand side (15) is also bounded from above by some constant, i.e., $E\|D_t\|^2 \leq c_U$.

The relation in (b) follows from the application of (9b) of Lemma 1 in the estimation of the fundamental matrix $\Phi_{A^*}(t,s)$ in expression (14).

Note 2. In case $\|G\| = 0$ the estimates for D_t may be slightly different than those given in Theorem 2. If $c_\phi < 2\kappa_2$, then matrix A_t^* is exponentially stable. Then, substituting $\|G\| = 0$ in (15), we obtain that $\|D_t\|^2 \to 0, t \to \infty$. Thus, under these conditions in the deterministic control system with discounting, the use of the optimal strategy in the long term ensures the achievement of the goal to stabilize the system. From relation (14) it follows that for $c_\phi \geq 2\kappa_2$ and $\|x - x_0\| > 0$ the exponential estimate of It. 2b of Theorem 2 remains unchanged.

The result of Theorem 2 makes it possible to analyze the consequences of the application of the optimal control obtained from the solution of the problem of stabilization of an economic system by using different types of discounting. The application of the asympotically nonpositive ensures that the mean square deviation of the trajectory of the process is kept within certain limits. At the same time, positive exponential discounting may lead to a considerable deviation of the system's state from its target level even in the absence of uncertainty. We show it by an example.

Example. Consider a deterministic linear regulator with exponential discounting. Assign the equation of state

$$dX_t = aX_t dt + U_t dt, \quad X_0 = x > 0,$$

and the cost functional

$$J_T(U) = \int_0^T \exp\{-\gamma t\}(qX_t^2 + U_t^2)dt,$$

where $a, q, \gamma > 0$ are constants; γ is the rate of discounting; and the target values are $x_0 = u_0 = 0$.

In the case of a system with constant parameters, the above-mentioned properties of uniform complete controllability and uniform complete observability of pairs of matrices, sufficient for the fulfillment of assumption $\mathcal{P}$, the equivalents of controllability $(a - \gamma/2, 1)$ and of observability are $(a - \gamma/2, q)$ (see [9]). As $B = 1 \neq 0$ and $Q = q > 0$, the considered linear control system has these properties and for it the assumption $\mathcal{P}$ is fulfilled. By Note 1 (see also the statement of Theorem 1), the optimal control is given as

$$U_t^* = -\Pi X_t^*,$$

where $\Pi > 0$ is the positive root of the Riccati algebraic equation

$$\Pi^2 - 2(a - \gamma/2)\Pi - q = 0. \tag{16}$$

The trajectory X_t^* dynamics is determined by the equation

$$dX_t^* = (a - \Pi)X_t^* dt, \quad X_0 = x.$$

We will show that there exist the values $a, q, \gamma > 0$ such that $a - \Pi > 0$, i.e., the trajectory X_t, instead of tending to zero (the target state $x_0 = 0$) will increase exponentially.

We find the positive root of Eq. (16):

$$\Pi = a - \gamma/2 + \sqrt{(a-\gamma/2)^2 + q}$$

and moreover $a - \Pi > 0$, if $\gamma/2 - \sqrt{(a-\gamma/2)^2 + q} > 0$. It is easy to see that the required condition is provided at $\gamma < a + q/a$.

Now we turn to the study of process $\|D_t\|^2$ in terms of the possibility of its estimation by means of the upper functions. As seen from the previous reasoning, not all kinds of discounting lead to the exponential stability of matrix A_t^*. We formulate and prove an auxilliary statement on the form of the upper function for the general case of the arbitrary bounded matrix in the equation of the process.

Lemma 2. *Let the stochastic process be given by the equation*

$$dS_t = \overline{A}_t S_t dt + G dw_t, \quad S_0 = \overline{s},$$

where $\overline{A}_t$ is the bounded time-varying matrix, G is the constant matrix, and $\overline{s}$ is a non-random vector. Then

(a) if matrix $\overline{A}_t$ *is exponentially stable, then the upper function for the process* $\|S_t\|^2$ *appears as* $h_t = \overline{c}\ln t$;

(b) *in other cases the upper function for the process* $\|S_t\|^2$ *has the order of an exponent, i.e.,* $h_t = \overline{c}\exp\{2\mu t\}$, *where* $\overline{c}, \mu > 0$ *are some positive constants and constant* μ *is chosen from the condition of the exponential stability of matrix* $\overline{A}_t - \mu \cdot I$.

Proof. The statement of (a) was proved in [8, Lemma A.2]. In other cases, when $\overline{A}_t$ is an arbitrary bounded matrix, there is a constant $\mu > 0$, providing the exponential stability of matrix $\overline{A}_t - \mu \cdot I$. In fact, by Lyapunov's estimate [13, p. 132], we have

$$\|\Phi_{\overline{A}}(t,s)\| \le c\exp\left\{\int_s^t \|\overline{A}_v\| dv\right\}, \quad 0 \le s \le t,$$

where $c > 0$ is a constant. By choosing $\mu > \sup_{t\ge 0}\|\overline{A}_t\|$, it is possible to achieve the exponential stability of matrix $\overline{A}_t - \mu \cdot I$.

Consider the process $Z_t = S_t/\exp\{\mu t\}$, described by the equation

$$dZ_t = (\overline{A}_t - \mu \cdot I)Z_t dt + G_t dw_t \quad Z_0 = \overline{s},$$

where $G_t = G\exp\{-\mu t\}$.

Due to the exponential stability of the $\overline{A}_t - \mu \cdot I$, matrix, for the process Z_t, the statement of Theorem 1 from [10] on the asymptotic behavior of the normalized solution of the linear stochastic equation holds. Specifically, in [10] it was shown that $\|Z_t\|^2 \big/ \int_0^t \|G_s\|^2 ds \to 0$ is almost surely at $t \to \infty$. As the normalizing function $\int_0^t \|G_s\|^2 ds = \|G\| \int_0^t \exp\{-2\mu t\}dt$ is bounded at any $t > 0$, then $\|Z_t\|^2 = \|S_t\|^2/\exp\{2\mu t\} \to 0$ is almost surely at $t \to \infty$. Therefore, the upper function for the process $\|S_t\|^2$ is $h_t = \overline{c}\exp\{2\mu t\}$ with constant $\overline{c} > 0$.

Theorem 3. *Let the conditions of Theorem* 1 *be fulfilled and* κ_2 *be the constant from* (5). *Then the upper function* h_t *for process* $\left\|X_t^* - x_0\right\|^2$ *is*

(a) *logarithmic function* $h_t = c_H \cdot \ln t$, *if* $c_\phi < 2\kappa_2$;

(b) *exponential function* $h_t = c_H \cdot \exp\{2\mu t\}$, *if* $c_\phi \ge 2\kappa_2$,

where $\mu > c_\phi/2 - \kappa_2, c_H > 0$ are some positive constants.

Proof. The assertion for the upper estimate in (à) follows from the sequential application of the statements of (a) in lemmas 1 and 2, determining in this the process $S_t := D_t = X_t - x_0$. To prove the form of the upper function in (b) note that by choosing $\mu > \mu_2$, where the constant $\mu_2 > (1/2)c_\phi - \kappa_2$ is taken from estimate (9) of (b) of Lemma 1, we obtain the exponential stability of matrix $A_t^* - \mu \cdot I$. Then, according to Lemma 2, the upper function for the process $\|S_t\|^2 = \left\|X_t^* - x_0\right\|^2$ is $h_t = c_H \cdot \exp\{2\mu t\}$ at constant $c_H > 0$.

The statement of Theorem 3 allows us to consider the effect of time preferences on the probabilistic properties of the average optimal control. If discounting is performed at an asymptotically nonpositive rate, then the corresponding process is upper bounded by the logarithmic function. The conventional exponential discounting may lead to a substantially different probabilistic properties of the average optimal control. First of all, it is possible that the exponential stability in the matrix in the linear equation of the process will be violated. Due to this there appear the upper estimates for the process, increasing much faster than the logarithm, which is the common bound in the case of constant parameters of the perturbations and the exponential stability of the mentioned matrix.

4. CONCLUSIONS

In the considered model, the time preferences of the subjects have a noticeable effect on the results of the dynamic optimization made in order to stabilize the system. Here, it is shown that the type of average optimality over an infinite-time horizon for the control law and the dynamics of the process corresponding to it are determined depending on the discounting function reflecting the time preferences. We note the main difference between the conventional exponential and other kinds of discounting lies in the fact that the use of the exponential discounting function in estimating future losses fails to guarantee the stabilization of optimal trajectories in the long run.

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