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On Stochastic Optimality for a Linear Controller with Attenuating Disturbances

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Abstract—For a linear stochastic control system with quadratic objective functional, we introduce various generalizations of the notions of optimality on average and stochastic optimality on an infinite time interval that take into account possible degeneration of the parameter of the disturbing process with time (attenuation of the disturbances) or the presence of a discount function in the objective functional. This lets us improve upon the quality estimate for a well known optimal control in this problem from the point of view of both asymptotic behavior of the functional's expectation and its asymptotic probabilistic properties. In particular, in the considered case we have found an improvement for the well known logarithmic upper bound on the optimal control for a family of defect processes.

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1. INTRODUCTION: MODEL DESCRIPTION AND DIFFERENT OPTIMALITY DEFINITIONS

This work is devoted to studying stochastic optimality in the control problem for a linear system with quadratic objective functional (see the linear controller problem in [1]). Traditional notions of optimality for controls in stochastic processes are related to the values of expectations of objective functionals defined on the trajectories of these processes. In what follows we call solutions of the corresponding optimization problems controls optimal on average. In the linear controller problem, such controls are well known for both finite and infinite time intervals (when optimality definitions are related to the asymptotic behavior of the objective functional's expectation); see, e.g., [2].

Unlike the traditional approach, definitions of stochastic optimality usually stem from criteria based on studying the asymptotic behavior of the functionals themselves in some probabilistic sense of the word. The main tool for studying stochastic optimality are various martingale versions of the laws of large numbers and limit theorems of probability theory. According to them, different kinds of optimality can be distinguished: optimality in probability, optimality almost surely (a.s.), optimality in distribution and so on. Brief surveys of this topic can be found in [3]; see also [4], where a result on stochastic optimality for linear controllers has been established that cannot be improved under standard assumptions on system parameters. These assumptions, which for linear controllable systems relate to such notions as exponential stability, stabilizability, controllability, recoverability and so on, in different combinations yield sufficient conditions for the existence of a so-called established optimal control law on an infinite time interval [2]. However, as we will see in this work, in certain special cases that have important applications the result of [4] can be significantly improved.

Below we give the model description and different optimality definitions that will be used throughout the paper. In particular, we introduce a new definition of optimality on average on an infinite time interval that generalizes preexisting definitions and lets us improve upon known results in the considered cases.

Consider a full probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ and an n-dimensional random process X_t, $t \geqslant 0$, defined in this space with the following equation:

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dw_t, \qquad X_0 = x, \tag{1}$$

where the initial state x is nonrandom; w_t, $t \geqslant 0$, a d-dimensional standard Wiener process; U_t, $t \geqslant 0$, an admissible control, i.e., a k-dimensional random process compatible with the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_t = \sigma\{w_s,\, s \leqslant t\}$ such that Eq. (1) has a solution; A_t, B_t, G_t, $t \geqslant 0$, bounded matrix functions of time with dimensions suitable for (1).

We denote the set of admissible controls by $\mathcal{U}$. For every $T > 0$ as the objective functional we define a random variable (r.v.)

$$J_T(U^T) = \int\limits_0^T (X_t' Q_t X_t + U_t' R_t U_t)\, dt, \tag{2}$$

where $U^T = \{U_t\}_{t \leqslant T}$ is a restriction of control $U \in \mathcal{U}$ on the interval $[0, T]$; Q_t, R_t, $t \geqslant 0$, bounded matrix functions of time that are nonnegative definite and positive definite respectively ($'$ denotes transposition). Here $U^T \in \mathcal{U}^T$, where $\mathcal{U}^T$ is the set of admissible controls considered on the interval $[0, T]$.

In the traditional sense, a control U^{*T} is called optimal on interval $[0, T]$ (we further call it optimal on average) if

$$EJ_T(U^{*T}) = \inf_{U^T \in \mathcal{U}^T} EJ_T(U^T). \tag{3}$$

For the model (1), (2), the form of a solution for problem (3) is well known (see, e.g., [2]). It is also known that under certain conditions there exists a control U^*, defined on an infinite time interval, that can be obtained by, in a certain sense, passing to the limit for U^{*T} as $T \to \infty$ (see [2]). As the planning horizon tends to infinity, one traditionally poses the optimality problem for U^* in the sense of averages in the long run, i.e., as a solution of the following problem:

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{T} \to \inf_{U \in \mathcal{U}}. \tag{4}$$

In case when the disturbance is degenerating as $t \to \infty$ or is attenuating ("small noise"), i.e., if $\|G_t\| \to 0$, where $\|\cdot\|$ denotes the Euclidean norm, criterion (4) turns out to be inefficient. Indeed, value of the quality functional on U^* can be written as $EJ_T(U^*) = \int_0^T tr(\Pi_t G_t G_t')\, dt + l_T$, where l_t is some bounded function and Π_t is a nonnegative definite bounded matrix function (see, e.g., [2]). If the parameters of control system (1), (2) are constant, in particular, $G_t \equiv G$, then $\Pi_t \equiv \Pi$, the criterion value in (4) on control U^* equals $tr(\Pi G G')$, and this value is minimal in the set of admissible controls. However, if $\|G_t\| \to 0$, $t \to \infty$, then the value of $EJ_T(U^*)$ may grow much slower than T (and may even be bounded if $\int_0^\infty \|G_t\|^2\, dt < \infty$), and in this case the criterion value in (4) turns out to be zero for the entire set of controls.

In another situation, even if disturbances are not attenuating but functions Q_t, R_t are infinitesimal for $t \to \infty$ (which happens in the case of a discount function in the functional that we consider in Section 5), it does not make sense to use the criterion from (4), and one must use a different definition for optimality on average on an infinite time interval. This definition for controllable diffusive processes, that also takes into account how discount function influences the behavior of the objective functional's expectation, has been given in [5]. Note also that in the special case of an exponential discount it is natural to pose a problem similar to (3) but considered not for a finite interval $[0, T]$ but rather for the interval $[0, \infty)$. As we will show, for $n = 1$ the problem with discount can be reduced to a standard linear controller problem with a suitable change of variables but with an attenuating disturbance. The situation of a degenerate disturbance also arises by itself in a number of applications (see, e.g., [6]).

In this work, we propose to use a new definition of optimality in the sense of long-run averages that generalizes the previous definition and lets us account for possible attenuation in the disturbing process. Besides, this definition will also imply the definition of optimality given in [5] for a problem with discount, a property demonstrated in this work with the help of the interrelation between problems that we have already noted.

In what follows we will assume that the system is not deterministic on the entire infinite time interval, that is, formally speaking, that $\int_0^\infty \|G_t\|^2\,dt > 0$.

Definition 1. A control $U^* \in \mathcal{U}$ is called *optimal on average on an infinite time interval* (or optimal in the sense of long-run averages) if it is a solution to the following problem:

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int\limits_0^T \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}. \tag{5}$$

Obviously, in case of constant parameters problems (4) and (5) are equivalent. At the same time, the functional in (5) directly takes into account the contribution of noise collected over the planning period. More precisely, it includes information about the ratios of expected costs (values of the functional) to the sum of variances of the random variables that form the integral noise vector $\mathcal{Z}_T = \int_0^T G_t\,dw_t$ on the interval $[0,T]$ (indeed, it is easy to show with the multidimensional Ito formula that $E\mathcal{Z}'_T\mathcal{Z}_T = \int_0^T \|G_t\|^2\,dt$).

Another optimality definition, the so-called "*overtaking*" optimality on average, has also been introduced in [5].

Definition 2. A control $U^* \in \mathcal{U}$ is called overtaking optimal on average if for every $\epsilon > 0$ there exists $T_0 > 0$ such that for an arbitrary admissible control $U \in \mathcal{U}$ it holds that

$$EJ_T(U^*) < EJ_T(U) + \epsilon \quad \text{for every} \quad T > T_0. \tag{6}$$

Note that overtaking optimality on average for a control implies its optimality in the sense of (5).

Studies of stochastic optimality are usually based on studying the asymptotic probabilistic behavior of the functional and comparing it for different controls: the control U^* defined as optimal on average and an arbitrary admissible control. For further definitions of stochastic optimality, we will use the notion of a defect process for control U^* (see [4]).

Definition 3. The defect process for a control U^* on control $U \in \mathcal{U}$ is the process

$$\Delta_T(U) := J_T(U^*) - J_T(U), \quad T > 0. \tag{7}$$

Choosing all possible $U \in \mathcal{U}$, we arrive at a family of processes $\{\Delta_T(U)\}_{U\in\mathcal{U}}$.

We give two definitions that let us compare the order of changes in the defect process in the sense of expectation and in the probabilistic sense with a certain positive nonincreasing function g_T (definitions of "g-optimality almost surely" and "g-optimality in probability" were first given in [7]).

Definition 4. A control U^* is called g-optimal on average if

$$\limsup_{T\to\infty} g_T E\Delta_T(U) \leqslant 0 \quad \text{for every} \quad U \in \mathcal{U}. \tag{8}$$

Definition 5. A control U^* is called g-optimal almost surely if

$$\limsup_{T\to\infty} g_T \Delta_T(U) \leqslant 0 \quad \text{with probability 1 for every} \quad U \in \mathcal{U}. \tag{9}$$

It is clear that in order to find a possible order of function g for which (9) holds it suffices to find an upper function for the family of defect processes defined as follows.

Definition 6. A nondecreasing function h_T is an *upper function for a family of defect processes* $\{\Delta_T(U)\}_{U\in\mathcal{U}}$ if for every $U \in \mathcal{U}$ there almost surely exists a finite time moment T_0 for which $\Delta_T(U) \leqslant h_T$ almost surely for $T > T_0$.

Once we have found an upper function h_T, g-optimality of control U^* will follow for $g_T = o(1/h_T)$. For the considered stochastic linear controller problem under standard assumptions on system parameters, in [4] we have obtained an upper function of the form $b_0 \ln T$, where $b_0 > 0$ is a constant.

The main purpose of this work is to get results on upper functions in the linear stochastic control problem that would generalize the results of [4], taking into account possible degeneration of function G_t for $t \to \infty$ and improving the order of upper functions in such cases. A new approach to defining g-optimality, related to finding such upper functions, can be characterized as an approach that uses as normalizing functions g_T functions of not only time T but also of the integral expression $\int_0^T \|G_t\|^2\,dt$ (there is a similarity here with different normalizations in (4) and (5) in defining optimality in the sense of long-run averages).

The paper is organized as follows. Section 2 contains basic assumptions of our model. Section 3 gives the form of a control U^* for which we formulate the theorem that it is, under corresponding assumptions, optimal in the sense of three different definitions: a new definition of averages in the long run (5), overtaking optimality on average, and g-optimality on average. Section 4 is devoted to studying probabilistic properties of this control. It contains statements related to the form of upper functions that imply, in particular, that in the case of a degenerate disturbance the defect process bound can be significantly improved compared to the logarithmic function found in [4]. Besides, in the same section we study stochastic optimality of control U^* from the point of view of one more definition: long-run averages with probability one. Section 5 considers the linear controller problem with discounting as it is used in economic, ecological, and behavioral applications; we show how it can be transformed to a standard problem with attenuating disturbance and apply our results to this model. Proofs of all statements are relegated to the Appendix.

2. BASIC ASSUMPTIONS OF OUR MODEL

For brevity we do not show here the form of a well known control in the synthesis form that represents a solution of problem (3); it can be found, e.g., in [2]. In this work, we consider a control defined on an infinite time interval, or the so-called established optimal control law U^* for $T \to \infty$. It also has a synthesis form (see [2]). Before we show the form of this control and formulate statements regarding its optimality in one sense or another, let us recount the assumptions introduced in [4] that are necessary for subsequent proofs. Conditions on the parameters of system (1), (2) that are sufficient for these assumptions to hold, as well as for the control U^* to exist, can be found in [4]. These are conditions standard for a linear controller considered on an infinite time interval, such as uniform complete controllability of the pair (A_t, B_t) and uniform complete recoverability of the pair (A_t, Q_t) or an exponential bound on the fundamental matrix for function A_t; see also [2].

We remind that the fundamental matrix $\Phi(t,s)$ for a matrix function $\mathcal{A}_t$, $t \geqslant 0$, is a solution of the following problem:

$$\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s), \quad \Phi(s,s) = I, \tag{10}$$

where I denotes the unit matrix. Here $\Phi(t,s) = \Phi(t,0)\Phi(0,s)$, $\Phi(s,t) = \Phi^{-1}(t,s)$.

Assumption 1. Functions A_t, B_t, Q_t, R_t, $t \geqslant 0$, are such that there exists an absolutely continuous bounded function Π_t, $t \geqslant 0$, with values in the set of nonnegative definite symmetric matrices that satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t'\Pi_t - \Pi_t B_t R_t^{-1} B_t'\Pi_t + Q_t = 0 \tag{11}$$

and such that the fundamental matrix $\Phi_{\mathcal{A}}(t,s)$ for function $\mathcal{A}_t := A_t - B_t R_t^{-1} B_t'\Pi_t$ satisfies the exponential bound

$$\|\Phi_{\mathcal{A}}(t,s)\| \leqslant \kappa_1 e^{-\kappa_2(t-s)}, \qquad s \leqslant t, \tag{12}$$

for some positive constants κ_1, $\kappa_2 > 0$.

Assumption 2. There exists a constant $c_0 > 0$ such that for every pair $(x_t, u_t)_{t\leqslant T}$ satisfying equation

$$dx_t = A_t x_t dt + B_t u_t dt, \quad x_0 = 0, \tag{13}$$

it holds that

$$\|x_T\|^2 + \int_0^T \|x_t\|^2 \, dt \leqslant c_0 \int_0^T (x_t' Q_t x_t + u_t' R_t u_t) dt. \tag{14}$$

In the next section, we give the form of a control for which it is known that under certain standard conditions on parameters it represents a solution of problem (4). We will show that in certain cases, it may also have stronger properties in the sense of optimality on average on an infinite time interval.

3. A CONTROL OPTIMAL ON AVERAGE ON AN INFINITE TIME INTERVAL

As we have already mentioned, in case of a linear–quadratic controller (under certain conditions on system parameters) solution of the control problem for $T \to \infty$ can be found as the limit of the solution of (3) for a finite T.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then*

(a) *a control of the form*

$$U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*, \tag{15}$$

where process $\{X_t^*\}_{t=0}^{\infty}$ *is defined by*

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dw_t, \quad X_0^* = x, \tag{16}$$

is a solution for problem (5);

(b) *control* U^* *is g-optimal on average for every function* g_T *such that* $g_T \to 0$, $T \to \infty$;

(c) *moreover, if at least one of the following two conditions holds:*

$$\int_0^\infty \|G_t\|^2 \, dt < \infty, \quad \lim_{t\to\infty} \|G_t\| = 0, \tag{17}$$

control U^* *is also overtaking optimal on average and g-optimal on average for* $g_T \equiv 1$.

To prove this theorem (the proof is given in the Appendix), we use a lemma that is interesting in its own right.

Lemma. *Suppose that Assumption 1 holds. Then* $E\|X_T^*\|^2$ *is bounded in* T. *If, moreover, at least one condition in* (17) *also holds then* $\lim_{T\to\infty} E\|X_T^*\|^2 = 0$.

The proofs of this lemma and all subsequent theorems are given in the Appendix.

4. PROBABILISTIC PROPERTIES OF CONTROLS OPTIMAL ON AVERAGE

In what follows we will use the notion of an upper function for an arbitrary process $\{Y_t\}_{t=0}^{\infty}$ defined on the probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ with filtration $\mathbf{F} = (\mathcal{F})_{t\geqslant 0}$.

Definition 7. A nondecreasing function h_T^* is an *upper function for process* $\{Y_T\}_{T\geqslant 0}$ if there almost surely exists a finite time moment $t_0(\omega)$ such that inequality $Y_T \leqslant h_T^*$ holds with probability one for every $T > t_0$.

We denote

$$\alpha_T := e^{-2\kappa_2 T} \int_0^T e^{2\kappa_2 t} \|G_t\|^2 \, dt. \tag{18}$$

Theorem 2. *Suppose that Assumption 1 holds. Then*

$$h_T^* = c_\alpha \sup_{t\leqslant T} (\alpha_t \ln t), \tag{19}$$

where $c_\alpha > 0$ is a certain constant, is an upper function for process $\|X_T^*\|^2$, where $\{X_t^*\}_{t=0}^{\infty}$ is given by Eq. (16). *Moreover, if $\lim_{T\to\infty} \alpha_T \ln T = 0$ then $\|X_T^*\|^2 \to 0$ with probability one, and any positive constant is an upper function for this process.*

Remark 1. An obvious corollary of Theorem 2 is the fact that if a function $\alpha_T \ln T$ is bounded for $T \to \infty$, then a certain constant $c^* > 0$ is also an upper function for the process $\|X_T^*\|^2$. Another obvious corollary of this theorem is a result obtained in [4].

In many natural situations, conditions of the theorem and the form of the upper function that use the value α_T defined in (18) can be restated in original terms that explicitly account for the behavior of the disturbing process parameter. We formulate the corresponding statements as corollaries.

Corollary 1. *Suppose that Assumption 1 holds. If $\lim_{t\to\infty} \|G_t\| = 0$ then an upper function of the process $Y_T = \|X_T^*\|^2$ always satisfies $h_T^* = o(\ln T)$.*

Corollary 2. *Suppose that Assumption 1 holds.*

(1) *If a function $\|G_T\|^2 \ln T$ is bounded for $T \to \infty$, then a certain positive constant is an upper function for process $\|X_T^*\|^2$. Moreover, if $\lim_{T\to\infty} \|G_T\|^2 \ln T = 0$ then $\|X_T^*\|^2 \to 0$ a.s., and any positive constant represents an upper function.*

(2) *Let $\|G_T\|^2 \ln T \to +\infty$ for $T \to \infty$ and suppose that starting from some finite moment of time, the function $\|G_t\|$ is differentiable, and it holds that*

$$-\frac{d}{dt} \ln \|G_t\| \leqslant \kappa_2 - \epsilon \tag{20}$$

for some $\epsilon > 0$. Then

$$h_T^* = c_G \sup_{t\leqslant T} (\|G_t\|^2 \ln t), \tag{21}$$

where $c_G > 0$ is a certain constant, is an upper function for process $\|X_T^*\|^2$.

Let us now turn to the problem of upper functions for the family of defect processes $\{\Delta_T(U)\}_{U\in\mathcal{U}}$ defined for a control (15).

Theorem 3. *Suppose that conditions of Theorem 1 hold, and $\hat{h}_T$ is an arbitrary nondecreasing unbounded function. Then a function of the form*

$$h_T = \max\{\hat{h}_T, h_T^*\}, \tag{22}$$

where h_T^* is an upper function for process $\|X_T^*\|^2$, defined in Theorem 2 (Corollary 2), under the corresponding assumptions is an upper function for the family of defect processes $\{\Delta_T(U)\}_{U\in\mathcal{U}}$.

In conclusion of this section, let us consider the problem of optimality for the control U^* in the sense of *stochastic long-run averages*, namely in the sense of solving the following problem:

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \quad \text{a.s.} \tag{23}$$

Let us formulate a few assumptions on the properties of function $\|G_t\|$.

Assumption 3. $\int_0^T \|G_t\|^2\,dt \to \infty$ for $T\to\infty$.

Assumption 4. Either the function $\|G_t\|^2 \ln t$ is bounded or $\|G_t\|^2 \ln t \to \infty$ for $t\to\infty$.

In some applications (see below an example of a linear–quadratic controller with generalized discounting) it is convenient to use the following assumption.

Assumption 5. The function $\|G_t\|$ is nonincreasing and differentiable.

Theorem 4. *Suppose that conditions of Theorem 1, Assumption 3, and either Assumption 4 or Assumption 5 hold. Then control U^* given by* (15) and (16) *is a solution of problem* (23).

Next we consider examples in which we will suppose that Assumptions 1 and 2 hold (and, consequently, statements (a) and (b) of Theorem 1 also hold).

Example 1. Let $\|G_t\| = \frac{2+\sin t}{(t+1)^\alpha}$, $\alpha > 0$. Then, obviously, statement (c) of Theorem 1 on overtaking optimality on average and g-optimality on average for $g_T \equiv 1$ hold for the control U^*. Since $\lim_{T\to\infty} \|G_T\|^2 \ln T = 0$, according to Corollary 2 an arbitrary positive constant will yield an upper function for the process $\|X_T^*\|^2$. Then, by Theorem 3, any increasing unbounded function is an upper function for the family of defect processes for control U^*. Further, if $\alpha \leqslant 1/2$ then $\int_0^T \|G_t\|^2\,dt \to \infty$, $T\to\infty$, so by Theorem 4 the control U^* will also be optimal in the sense of solving problem (23).

Example 2. Let $\|G_t\| = \frac{1}{\ln^\beta(t+2)}$, $0 < \beta < 1/2$. Then statement (c) of Theorem 1 also holds; by Corollary 2, function $h_T^* = \frac{c_G \ln T}{\ln^{2\beta}(T+2)}$ is an upper function for process $\|X_T^*\|^2$, so, consequently, function $h_T = \bar{c}(\ln T)^{1-2\beta}$ for some $\bar{c}$ is also such, and by Theorem 3 it is also an upper function for the defect process. Besides, it is easy to see that function $\|G_t\|$ satisfies Assumptions 3 and 4, so the statement of Theorem 4 also holds.

We consider one more sample application of our results in the next section.

5. A LINEAR–QUADRATIC CONTROLLER WITH GENERALIZED DISCOUNTING

Consider a linear controllable system of the following form. A one-dimensional random process $\{\hat{X}_t\}_{t=0}^\infty$ is defined over a probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ with the following equation:

$$d\hat{X}_t = a\hat{X}_t dt + b\hat{U}_t dt + \sigma dw_t, \qquad \hat{X}_0 = x, \tag{24}$$

where $\{w_t\}_{t=0}^\infty$ is a one-dimensional standard Wiener process; a, $b \neq 0$, $\sigma > 0$ are constants; $\{\hat{U}_t\}_{t=0}^\infty$, a control admissible in the sense described in Section 1. Consider the deterministic process given by Eq. (24) for $\sigma = 0$, and let x_0 be a fixed number. Then $\hat{X}_0 \equiv x_0$ is the stationary state of this process for control $\hat{U}_t \equiv u_0 = -\frac{a}{b}x_0$. We define the objective functional that accounts for losses due to the deviation of the random (perturbed) process and a certain admissible control from x_0 and u_0 respectively:

$$J_T(\hat{U}^T) = \int_0^T f_t(q(\hat{X}_t - x_0)^2 + (\hat{U}_t - u_0)^2)\,dt, \tag{25}$$

where $q > 0$ is a constant; f_t, a discounting function. We assume that function f_t has the following properties: (1) $f_0 = 1$, $f_t > 0$ for $t \geqslant 0$; (2) f_t is nonincreasing and differentiable on $[0,\infty)$; (3) the decrease rate of function f_t is bounded, i.e., there exists $K > 0$ such that $-\frac{d}{dt}\ln f_t \leqslant K$. For instance, for $f_t \equiv 1$ the model (24) and (25) corresponds to a standard stochastic linear–quadratic controller. If $f_t = e^{-\gamma t}$ ($\gamma > 0$) then it is a problem with standard discounting; for $f_t = \frac{1}{(1+\theta t)^{\theta_1/\theta}}$ ($\theta_1, \theta > 0$) we get the general form of a hyperbolic discounting problem [8]; in case when $f_t = m_1 e^{-\alpha t} + (1-m_1)e^{-\beta t}$ ($\alpha, \beta > 0$, $0 < m_1 < 1$) we get a problem with double discounting [9]. Such discount factors have been traditionally used in ecological, economic, and behavioral models [8–10].

In [5], the author studied a control problem for a diffusive process with quality criterion $\liminf_{T\to\infty}\left(E\int_0^T f_t c(x_t,u_t)\,dt/\int_0^T f_t\,dt\right)$, where $c(x,u)$ is a certain function and the discounting function f_t is such that $f_T \to 0$, $\int_0^T f_t\,dt \to \infty$ as $T \to \infty$. In particular, existence conditions for overtaking optimal on average controls were obtained.

In case $\lim_{t\to\infty} f_t = 0$, system (24) and (25) does not have a property belonging to the system of sufficient existence conditions for an established optimal control law on an infinite time interval, namely the property that coefficient f_t in the second term of functional (25) is separated from zero (see [2]). However, we will see that by a change of variables this problem can be reduced to the problem of a standard stochastic linear–quadratic controller for which the said sufficient conditions do hold. And then the problem begins to conform to the results found in this work. Thus, we denote

$$X_t := \sqrt{f_t}(\hat{X}_t - x_0), \quad U_t := \sqrt{f_t}(\hat{U}_t - u_0). \tag{26}$$

Dynamics of the process $\{X_t\}$ will be given by equation

$$dX_t = a_t X_t dt + bU_t dt + \sigma_t dw_t, \qquad X_0 = x - x_0, \tag{27}$$

where $2a_t = 2a + \dot{f}_t/f_t$ is a bounded function, $\sigma_t = \sigma\sqrt{f_t}$.

The functional (25) in new notation assumes the following form:

$$J_T(U) = \int_0^T (qX_t^2 + U_t^2)\,dt. \tag{28}$$

The problem of the form (5) for finding a control optimal on average on an infinite time interval is now formulated as follows:

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T \sigma_t^2\,dt} = \limsup_{T\to\infty} \frac{EJ_T(U)}{\sigma^2\int_0^T f_t\,dt} \to \inf_{U\in\mathcal{U}}. \tag{29}$$

It is easy to check (see [10]) that system (27)–(29) satisfies Assumptions 1 and 2, so, consequently, by Theorem 1 a control U^* of the form (15) is a solution for problem (29). To find the character of the upper function we can use Corollary 2 and Theorem 3. For instance, for cases of standard, double exponential, and hyperbolic discounting we get as a result that any nondecreasing unbounded function can serve as the upper function for the defect process of control U^*. For classical discounting it is not an unexpected result since on the optimal control the functional (28) as $T \to \infty$ asymptotically converges to a certain random value, and the worst possible result in the sense of the defect process can be found with a competing control on which (28) asymptotically converges to a different random value. In the case of a hyperbolic discount factor with $\theta_1 \leqslant \theta$, the functional (28) on U^* will already tend to infinity for $T \to \infty$. In this case, the possibility to upper bound the family of defect processes with any function $h_T \to \infty$, $T \to \infty$ means that the difference of the values of functionals may grow arbitrarily slow.

Since $\sigma_t = \sigma\sqrt{f_t}$ is nonincreasing and differentiable, it means that for the case when $\int_0^T f_t dt \to \infty$, $T \to \infty$ the statement of Theorem 4 also holds, i.e., U^* is stochastically optimal in the sense of (23). For instance, it holds for a hyperbolic discount function for $\theta_1 \leqslant \theta$.

6. CONCLUSION

This work is devoted to studying stochastic optimality in dynamical systems, a problem that has become especially relevant lately due to increased interest in studying behavior objective functionals on optimal trajectories in one probabilistic sense or another. The notion of stochastic optimality based on asymptotic probabilistic comparison of objective functionals for different controls on large time intervals turns out to be stronger than traditional optimality on average since it lets us make judgements about the control quality not only on average for all realizations but also on a single individual trajectory.

If we study systems considered with infinite planning horizons, then from the point of view of both optimality on average and stochastic optimality it is natural to study controls that are established (as the planning horizon tends to infinity) optimal on average controls. In the linear–quadratic controller problem, the established optimal control law that exists under certain conditions on the parameters is well known, and it also is optimal on average in the sense of long-run averages and also in a certain stochastic sense if we apply to the functional (that is, to be more precise, to the defect process) a normalization in the form of a function that tends to infinity slightly faster than logarithmically. However, it turns out that in certain nonstandard situations, e.g., if the disturbing process parameter degenerates with time or the functional contains a discounting function that tends to zero, the usual long-run averages criterion becomes inefficient, and normalizations of the functional and defect process used in the general case can be significantly weakened.

In this work, we have proposed a generalization of the long-run averages criterion that accounts for the contribution of noise to system dynamics. As we studied the rate of growth for the defect process, we have obtained a general representation for upper functions in many cases, in particular for attenuating disturbances, that lets us improve over a previously known logarithmic bound. As a direction for further study, we single out the study of stochastic optimality in situations when disturbances increase in time, i.e., for $\|G_t\| \to \infty$, $t \to \infty$, and formulations of more general optimality conditions in the sense of stochastic long-run averages for a standard controller.

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APPENDIX

Proof of Lemma. Solution of Eq. (16) has the form

$$X_T^* = \Phi_{\mathcal{A}}(T,0)x + \int_0^T \Phi_{\mathcal{A}}(T,t)G_t\, dw_t, \tag{A.1}$$

where $\Phi_{\mathcal{A}}(t,s)$ is the fundamental matrix for function $\mathcal{A}_t := A_t - B_t R_t^{-1} B_t' \Pi_t$. Using the Ito's isometry property, we get that

$$E\|X_T^*\|^2 \leqslant \|\Phi_{\mathcal{A}}(T,0)\|^2\|x\|^2 + \int_0^T \|\Phi_{\mathcal{A}}(T,t)\|^2\|G_t\|^2\, dt.$$

In what follows we will denote by c all constants whose specific values do not matter and may change from formula to formula. Using the exponential bound (12) formulated in Assumption 1, we get

$$E\|X_T^*\|^2 \leqslant ce^{-2\kappa_2 T}\|x\|^2 + c\int\limits_0^T e^{-2\kappa_2(T-t)}\|G_t\|^2\,dt, \tag{A.2}$$

which, since function G_t is bounded, implies that $E\|X_T^*\|^2$ is bounded. Suppose now that the first condition from (17) holds. We show that in this case the second term in (A.2), just like the first, is also infinitesimal as T tends to infinity. To prove it, fix an $\varepsilon > 0$ and choose T_0 such that $\int_{T_0}^{\infty}\|G_t\|^2\,dt < \varepsilon$. Then

$$\int\limits_0^T e^{-2\kappa_2(T-t)}\|G_t\|^2\,dt \leqslant \int\limits_0^{T_0} e^{-2\kappa_2(T-t)}\|G_t\|^2\,dt + \int\limits_{T_0}^T \|G_t\|^2\,dt \leqslant ce^{-2\kappa_2 T} + \varepsilon,$$

and since ε was arbitrary we get the necessary bound. Suppose now that the second condition from (17) holds. We compute the limit of the second term in (A.2) with the l'Hôpital's rule: $\lim_{T\to\infty} e^{-2k_2T}\int_0^T e^{2k_2t}\|G_t\|^2\,dt = \lim_{T\to\infty}\|G_T\|^2/(2k_2) = 0$, i.e., again $\lim_{T\to\infty} E\|X_T^*\|^2 = 0$.

Proof of Theorem 1. Fix an arbitrary admissible control $U \in \mathcal{U}$, denote by X_t the corresponding phase process, and let us find the corresponding defect process (7) for control U^*. We denote $x_t := X_t - X_t^*$, $u_t := U_t - U_t^*$. Note that the pair $(x_t, u_t)_{t\leqslant T}$ satisfies (13). Under Assumptions 1 and 2, for the defect process $\Delta_T = \Delta_T(U)$ an estimate of the following form has been bound in [4]:

$$\Delta_T \leqslant c_1\|X_T^*\|^2 - c_2\int\limits_0^T \|x_t\|^2\,dt - 2\int\limits_0^T x_t'\Pi_t G_t dw_t, \tag{A.3}$$

where c_1, c_2 are some positive constants. Consequently,

$$J_T(U^*) \leqslant J_T(U) + c_1\|X_T^*\|^2 - 2\int\limits_0^T x_t'\Pi_t G_t dw_t. \tag{A.4}$$

This, obviously, implies that

$$EJ_T(U^*) \leqslant EJ_T(U) + c_1 E\|X_T^*\|^2, \tag{A.5}$$

and after normalization we get

$$\frac{EJ_T(U^*)}{\int_0^T \|G_t\|^2\,dt} \leqslant \frac{EJ_T(U)}{\int_0^T \|G_t\|^2\,dt} + c_1\frac{E\|X_T^*\|^2}{\int_0^T \|G_t\|^2\,dt}. \tag{A.6}$$

Therefore, if the integral $\int_0^\infty \|G_t\|^2\,dt$ diverges, we get from (A.6) that

$$\limsup_{T\to\infty}\frac{EJ_T(U^*)}{\int_0^T \|G_t\|^2\,dt} \leqslant \limsup_{T\to\infty}\frac{EJ_T(U)}{\int_0^T \|G_t\|^2\,dt}, \tag{A.7}$$

i.e., control U^* is a solution of problem (5). If, on the other hand, the said integral converges then by Lemma 1 $\lim_{T\to\infty} E\|X_T^*\|^2 = 0$, and then (A.7) will also hold. This proves statement (a) of the Theorem 1. Let us check for completeness that the expression in the left-hand side of (A.7) is finite. From (11) and (16) with the Ito's formula we get that $d((X_t^*)'\Pi_t X_t^*) = -\{(X_t^*)'Q_t X_t^* + (U_t^*)'R_t U_t^*\}dt + tr(G_t'\Pi_t G_t)dt + 2(X_t^*)'\Pi_t G_t dw_t$, i.e.,

$$J_T(U^*) = x'\Pi_0 x - (X_T^*)'\Pi_T X_T^* + \int_0^T tr(G_t'\Pi_t G_t)\, dt + 2\int_0^T (X_t^*)'\Pi_t G_t dw_t. \tag{A.8}$$

Consequently, $EJ_T(U^*) \leqslant x'\Pi_0 x + \int_0^T tr(G_t'\Pi_t G_t)\, dt$, and since functions Π_t, G_t are bounded we get that $\limsup_{T\to\infty}\left(EJ_T(U^*)/\int_0^T \|G_t\|^2\, dt\right) < \infty$.

The latter two statements of the theorem can be shown immediately with inequality (A.5) and Lemma 1, which concludes the proof of Theorem 1.

Proof of Theorem 2. Following an idea from [4] (see the proof of Lemma A.2 there), we first consider the process $\{\tilde X_t\}_{t=0}^\infty$ whose dynamics is given by equation

$$d\tilde X_t = -\kappa_2 I \tilde X_t\, dt + G_t dw_t, \quad \tilde X_0 = 0, \tag{A.9}$$

where I is the unit matrix. Solving (A.9) and making a standard change of time for each component in the solution $\tilde X_T^i$, $i = 1, \ldots, n$, we get

$$\tilde X_T^i = e^{-\kappa_2 T}\sum_{j=1}^d \int_0^T e^{\kappa_2 t} G_t^{ij} dw_t^j = e^{-\kappa_2 T} M_T^i = e^{-\kappa_2 T}\hat w^i_{\langle M_T^i\rangle}, \tag{A.10}$$

where $\hat w^i$ is a Wiener process, G_t^{ij} are elements of matrix G_t, and $\langle M_T^i\rangle = \int_0^T e^{2\kappa_2 t}\left(\sum_{j=1}^d (G_t^{ij})^2\right) dt$ is the quadratic characteristic of martingale M_T^i. Thus, it holds on the set $\{\langle M_\infty^i\rangle < \infty\}$ that $(\tilde X_T^i)^2 \to 0$ a.s. On the set $\{\langle M_T^i\rangle \to \infty\}$ we apply the repeated logarithm law to Wiener process $\hat w^i$ and conclude as a result that there exist a number $c > 0$ and a time moment $t_0^i(\omega)$ such that $(\hat w^i_{\langle M_T^i\rangle})^2 \leqslant c\langle M_T^i\rangle \ln\ln\langle M_T^i\rangle$ for $T > t_0^i$. Since elements of the matrix function G_t are bounded, we have $\ln\ln\langle M_T^i\rangle \leqslant c\ln T$. Therefore, for $T > t_0^i(\omega)$ it holds that

$$(\tilde X_T^i)^2 \leqslant ce^{-2\kappa_2 T}\int_0^T e^{2\kappa_2 t}\|G_t\|^2\, dt \ln T = c\alpha_T \ln T \quad \text{a.s.}, \quad i = 1, \ldots, n,$$

so

$$\|\tilde X_T\|^2 \leqslant \tilde c \alpha_T \ln T \tag{A.11}$$

with probability one for $T > \tilde t_0(\omega)$ for some $\tilde t_0 = \tilde t_0(\omega)$ and some constant $\tilde c > 0$.

Consider now the process $Z_t = X_t^* - \tilde X_t$, which due to (16) and (A.9) is given by relations $dZ_t = (A_t - B_t R_t^{-1} B_t'\Pi_t)Z_t\, dt + (A_t + \kappa_2 I)\tilde X_t\, dt$, $Z_0 = x$ and has, respectively, the form $Z_T = \Phi_{\mathcal A}(T, 0)x + \int_0^T \Phi_{\mathcal A}(T, t)(A_t + \kappa_2 I)\tilde X_t\, dt$, where $\Phi_{\mathcal A}(t, s)$ is the fundamental matrix for matrix function $\mathcal A_t := A_t - B_t R_t^{-1} B_t'\Pi_t$. Then for $T > \tilde t_0$ (where $\tilde t_0$ is the random time moment defined above)

$$\begin{aligned}\|Z_T\| \leqslant \|\Phi_{\mathcal A}(T, 0)x\| &+ \int_0^{\tilde t_0} \|\Phi_{\mathcal A}(T, t)\|\|(A_t + \kappa_2 I)\tilde X_t\|\, dt \\ &+ \int_{\tilde t_0}^T \|\Phi_{\mathcal A}(T, t)\|\|(A_t + \kappa_2 I)\|\|\tilde X_t\|\, dt.\end{aligned} \tag{A.12}$$

Since Assumption 1 holds, the first two terms in the right-hand side of the latter inequality tend to zero a.s. In order to estimate the last term, note that for the function $\tilde{h} = \tilde{c}\sup_{t\leqslant T}(\alpha_T \ln T)$

$$\tilde{h}_T^{-1/2}\int\limits_{\tilde{t}_0}^{T}\|\Phi_{\mathcal{A}}(T,t)\|\|(A_t+\kappa_2 I)\|\|\tilde{X}_t\|\,dt \leqslant \int\limits_{\tilde{t}_0}^{T}\|\Phi_{\mathcal{A}}(T,t)\|\|(A_t+\kappa_2 I)\|\tilde{h}_t^{-1/2}\|\tilde{X}_t\|\,dt,$$

and due to relation (A.11) the latter expression is bounded a.s. Thus, it easily follows from (A.11) by definition of the process Z_t that function h^* defined in (19) (and coinciding with $\tilde{h}$ up to a constant factor) is an upper function for the process $\|X_T^*\|^2$. If $\lim_{T\to\infty}\alpha_T \ln T = 0$ then, due to (A.11), $\|\tilde{X}_T\| \to 0$ a.s. Then it is easy to see that $\|Z_T\| \to 0$ a.s. as well, and this proves the last statement of Theorem 2.

Proof of Corollary 1. If $\alpha_T \ln T$ is a bounded function, the statement of this corollary is obvious. Otherwise, the integral in (18) must also be an unbounded growing function of T. Then (using, for instance, l'Hôpital's rule), it is easy to show that if $\lim_{t\to\infty}\|G_t\| = 0$ then also $\lim_{T\to\infty}\alpha_T = 0$. Since the continuous function $\alpha_T \ln T$ is unbounded, for every nondecreasing sequence $T_i > 0$, $T_i \to \infty$, there exists a nondecreasing sequence $\bar{T}_i > 0$, $\bar{T}_i \to \infty$, $\bar{T}_i \leqslant T_i$, for which it holds that $\sup_{t\leqslant T_i}(\alpha_t \ln t) = (\alpha_{\bar{T}_i}\ln \bar{T}_i)$. Consequently, $\sup_{t\leqslant T_i}(\alpha_t \ln t)/\ln T_i \leqslant \alpha_{\bar{T}_i}\ln\bar{T}_i/\ln\bar{T}_i = \alpha_{\bar{T}_i} \to 0$, $i\to\infty$. Since by Theorem 2 the upper function $h_T^* = c_\alpha \sup_{t\leqslant T}(\alpha_t \ln t)$, then, since the sequence T_i was arbitrary, we conclude that $h_T^* = o(\ln T)$.

Proof of Corollary 2. It is easy to see that if function $\|G_T\|^2 \ln T$ is bounded for $T \to \infty$, then function $\alpha_T \ln T$ is also bounded, and then statement from item (1) of Corollary 2 holds (see Remark 1 to Theorem 2). If at the same time $\lim_{T\to\infty}\|G_T\|^2 \ln T = 0$, we can show (using, in particular, l'Hôpital's rule) that $\lim_{T\to\infty}\alpha_T \ln T = \frac{1}{2\kappa_2}\lim_{T\to\infty}\|G_T\|^2 \ln T = 0$. The latter relation together with Theorem 2 concludes the proof of item (1).

To show the statement of item (2) it suffices to show that starting from a certain time moment it holds that $\alpha_t \leqslant c\|G_t\|^2$, where $c > 0$ is a certain constant. We denote $\phi_t = \alpha_t/\|G_t\|^2$, defined due to conditions of item (2) at least for sufficiently large t. It is easy to check that ϕ_t satisfies the linear differential equation $\frac{d\phi_t}{dt} = \left(-2\left(\kappa_2 + \frac{d}{dt}\ln\|G_t\|\right)\phi_t + 1\right)$, and in order for its solution to be bounded it suffices that (20) holds. Thus, $c_\alpha \sup_{t\leqslant T}(\alpha_t \ln t) \leqslant c_G \sup_{t\leqslant T}(\|G_t\|^2 \ln t)$ for some constant $c_G > 0$, which by Theorem 2 implies that statement (2) of the corollary holds.

Proof of Theorem 3. Using inequality (A.3) and taking into account the fact that functions Π_t and G_t are bounded, we get that for some c_1, c_2 it holds that

$$\Delta_T \leqslant c_1\|X_T^*\|^2 + \mathcal{R}_T, \tag{A.13}$$

where

$$\mathcal{R}_T := -c_2\int\limits_0^T \|G_t'\Pi_t x_t\|^2\,dt - 2\int\limits_0^T x_t'\Pi_t G_t dw_t. \tag{A.14}$$

According to the results of [4] (see Lemma A.1), the process $\mathcal{R}_T$ is a.s. bounded from above by some finite random variable, so every nondecreasing unbounded function $\hat{h}_T$ will be an upper function for the process $\mathcal{R}_T$. The upper function h_T^* for process $\|X_T^*\|^2$ was defined in Theorem 2 (Corollary 2). Therefore, a function dominating both processes in (A.13) will be an upper function for process $\Delta_T = \Delta_T(U)$, i.e., $h_T = \max\{\hat{h}_T, h_T^*\}$, and this concludes the proof of Theorem 3.

Proof of Theorem 4. It follows from (A.13) that

$$\frac{J_T(U^*)}{\int\limits_0^T\|G_t\|^2\,dt} \leqslant \frac{J_T(U)}{\int\limits_0^T\|G_t\|^2\,dt} + c_1\frac{\|X_T^*\|^2}{\int\limits_0^T\|G_T\|^2\,dt} + \frac{\mathcal{R}_T}{\int\limits_0^T\|G_t\|^2\,dt}. \tag{A.15}$$

Under the conditions of Assumption 3 and due to the properties of process $\mathcal{R}_T$ described above, the limit of the positive part of the last term in inequality (A.15) is a.s. zero. Further, suppose that Assumption 4 holds. If function $\|G_t\|^2 \ln t$ is bounded, then according to Corollary 2 some positive constant represents an upper function for $\|X_T^*\|^2$, and the limit of the second term in (A.15) is also zero a.s. If, on the other hand, $\|G_t\|^2 \ln t \to \infty$ for $t \to \infty$, we use the general form of an upper function for process $\|X_T^*\|^2$ found in [4]. Since $b_0 \ln T$ is such a function for some constant $b_0 > 0$, and, moreover, it is easy to see that under the said assumption $\lim_{T\to\infty} \ln T/\left(\int_0^T \|G_t\|^2\,dt\right) = \lim_{T\to\infty} 1/(T\|G_T\|^2) = 0$, we conclude again that the limit of the second term in (A.15) is zero a.s.

It remains to show that the latter takes place also in case Assumption 5 holds. We first show that a similar property holds for the process defined in (A.9), i.e.,

$$\|\tilde{X}_T\|^2 / \int\limits_0^T \|G_t\|^2\,dt \to 0 \text{ a.s.} \tag{A.16}$$

As we have seen above, estimate (A.11) holds for process $\|\tilde{X}_T\|^2$. At the same time, under the conditions of Assumption 3 it is easy to check that

$$\begin{aligned} &\lim_{T\to\infty} \alpha_T \ln T / \int\limits_0^T \|G_t\|^2\,dt \\ &= \frac{1}{2\kappa_2} \lim_{T\to\infty} (\|G_T\|^2 \ln T) / \left(\int\limits_0^T \|G_t\|^2\,dt + \|G_T\|^2 \right). \end{aligned} \tag{A.17}$$

If at the same time Assumption 5 holds, then for $T_0 : T > T_0 > 0$, integrating by parts, we can show that the following inequality also holds

$$\|G_T\|^2 \ln T \leqslant \int\limits_{T_0}^T \frac{\|G_t\|^2}{t}\,dt + \|G_{T_0}\|^2 \ln T_0.$$

Then (A.17) implies that $\lim_{T\to\infty} \alpha_T \ln T/\int_0^T \|G_t\|^2\,dt = 0$, and (A.16) holds. Further, using inequality (A.12) for process $Z_t = X_t^* - \tilde{X}_t$, we see that $\|Z_T\|^2 \int_0^T / \|G_t\|^2\,dt \to 0$ a.s. and, therefore, due to (A.16) a similar relation also holds for the process X_T^*. This concludes the proof of Theorem 4.

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