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Asymptotic Behavior of the Solution to a Linear Stochastic Differential Equation and Almost Sure Optimality for a Controlled Stochastic Process

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Abstract—The asymptotic behavior of a stochastic process satisfying a linear stochastic differential equation is analyzed. More specifically, the problem is solved of finding a normalizing function such that the normalized process tends to zero with probability 1. The explicit expression found for the function involves the parameters of the perturbing process, and the function itself has a simple interpretation. The solution of the indicated problem makes it possible to considerably improve almost sure optimality results for a stochastic linear regulator on an infinite time interval.

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1. INTRODUCTION

Linear stochastic differential equations are widely used in the simulation of physical, chemical–biological, economic, and other processes (see, e.g., [1–4]) and in the mathematical control theory [5, 6]. The study of the asymptotic behavior of solutions to such equations plays an important in various applications. For example, in [7], this issue was directly related to the possibility of applying the simulated annealing algorithm to global minimum search (see [1]). In [6, 8] information of this kind was also required in the study of almost sure optimality (i.e., optimality with probability 1) for a controlled stochastic process on an infinite horizon in various formulations (so-called stochastic optimality). In the study of stochastic optimality in the control problem for a linear system with a quadratic objective functional (linear-quadratic regulator), a key issue addressed in [6] was the almost sure asymptotic behavior of a controlled process with average-optimal control. In this paper, we examine more general asymptotic properties of this process, which makes it possible to improve and generalize the results obtained in [6, 8] concerning almost sure optimality in the sense of the so-called stochastic long-run average in the linear-quadratic regulator problem for various nonstandard situations, including a general discount function present in the cost functional.

Consider an n-dimensional stochastic process X_t, $t \geq 0$, defined on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. The dynamics of this process is described by the equation

$$dX_t = A_t X_t dt + G_t dw_t, \quad X_0 = x, \tag{1.1}$$

where the initial state x is nonrandom, w_t $(t \geq 0)$ is a d-dimensional standard Wiener process, and A_t and G_t $(t \geq 0)$ are bounded matrix functions of appropriate sizes. Specifically, an equation of form (1.1) describes the above controlled stochastic process with average-optimal control in the linear-quadratic regulator problem. Additionally, all the processes considered in what follows are assumed to be defined on the same probability space.

Assume that the solution of problem (1.1) with $G_t \equiv O$ (O is a zero matrix) and with the initial condition $X_{t_0} = x$ ($t_0 \geq 0$ is an arbitrary moment of time), i.e., the solution of a deterministic equation, is exponentially stable. The following assumption is made according to the definition of exponential stability (see, e.g., [5]).

Assumption 1. The function A_t is such that the fundamental matrix $\Phi_{\mathcal{A}}(t, s)$ for the function $\mathcal{A}_t := A_t$ admits the exponential estimate

$$\|\Phi_{\mathcal{A}}(t, s)\| \le \kappa_1 e^{-\kappa_2(t-s)}, \quad s \le t, \tag{1.2}$$

where $\kappa_1, \kappa_2 > 0$ are positive constants and $\|\cdot\|$ is the Euclidean norm. Recall that the fundamental matrix $\Phi_{\mathcal{A}}(t, s)$ for the matrix function $\mathcal{A}_t$, $t \ge 0$, solves the problem

$$\frac{\partial \Phi_{\mathcal{A}}(t, s)}{\partial t} = \mathcal{A}_t \Phi_{\mathcal{A}}(t, s), \quad \Phi_{\mathcal{A}}(s, s) = I, \tag{1.3}$$

where I is the identity matrix. Moreover, $\Phi_{\mathcal{A}}(t, s) = \Phi_{\mathcal{A}}(t, 0)\Phi_{\mathcal{A}}(0, s)$ and $\Phi_{\mathcal{A}}(s, t) = \Phi_{\mathcal{A}}^{-1}(t, s)$.

Two approaches can be used in the study of asymptotic probability properties of the solution to problem (1.1). One is associated with the construction of upper functions for processes. It is well known (see [6, 8]) that upper functions majorize a process with probability 1, starting at a finite time, and the search for them is based on applying the law of the iterated logarithm for stochastic integrals. In the general case of a bounded function G_t satisfying Assumption 1, it was shown in [6] that an upper function for $\|X_t\|^2$ is $b_0 \ln t$, where $b_0 > 0$ is a constant. The form of the upper function was refined in [8], where its various representations involving the parameters of the perturbing process (as functionals of $\|G_t\|^2$, $t \ge 0$) were obtained. The other approach involves the search for suitable normalizing multipliers guaranteeing that the process almost surely (a.s.) tends asymptotically to zero. This approach is associated with applying limit theorems from probability theory, specifically, the strong law of large numbers for semimartingales (see [9]). Examples of assertions of this type for solutions of stochastic differential equations can be found in [10]. In this paper, the result of [11] on the behavior of the process $\|X_t\|^2$ is extended to several dimensions in the case of suitably chosen normalization, which involves the parameters of the perturbing process in explicit form and has a simple interpretation. Accordingly, as was said above, the result of [8] on almost sure optimality in the linear regulator problem can be generalized without using a number of additional constraints.

This work is organized as follows. Section 2 presents the main result concerning the almost sure convergence of a normalized process and gives several auxiliary assertions that are of interest on their own. Additionally, an example of modeling a process is presented that illustrates the validity of the results proved. In Section 3, the results are applied to the study of stochastic optimality in the infinite-horizon linear regulator problem. A problem with discounting is considered separately.

2. MAIN RESULTS AND A MODELING EXAMPLE

2.1. Main Results

Theorem 1. *Let Assumption* 1 *hold and* $\int_0^\infty \|G_t\|^2 dt > 0$. *Then, for the process* X_t *described by Eq.* (1.1), *it is true that* $\|X_T\|^2 / \int_0^T \|G_t\|^2 dt \longrightarrow 0$ *a.s. as* $T \longrightarrow \infty$.

Note that the normalization $\int_0^T \|G_t\|^2 dt$ directly takes into account the contribution of the noise accumulated over the time T. More precisely, it is equal to the sum of the variances of the random variables making up the vector of integral disturbances $\mathfrak{X}_T = \int_0^T G_t dw_t$ on the interval $[0, T]$ (indeed, it is easy to show with the use of the multidimensional Ito formula that $E\mathfrak{X}_T'\mathfrak{X}_T = \int_0^T \|G_t\|^2 dt$).

To prove Theorem 1, we need several lemmas.

Lemma 1. *Let* $\{\xi_t\}_{t=0}^\infty$ *be a stochastic process,* $\{\tilde{w}_t\}_{t=0}^\infty$ *be a standard one-dimensional Wiener process, and the stochastic integral* $\int_0^T \xi_t d\tilde{w}_t$ *be defined. Suppose also that* Γ_t *is a nondecreasing function and* $\Gamma_0 = 0$. *Given any* a, b: $0 \le a < b$, *if* $\int_a^b E\xi_t^2 dt \le c_1(\Gamma_b - \Gamma_a)$ (*where* $c_1 > 0$ *is a constant*) *and* $\lim_{T \to \infty} \Gamma_T = \infty$, *then* $\int_0^T \xi_t d\tilde{w}_t / \Gamma_T \longrightarrow 0$ *a.s. as* $T \longrightarrow \infty$.

Proof. This result is proved using the approach described, for example, in [12]. It is based on well-known tests for the convergence of random sequences to discretized versions of continuous-time processes. Define the process $\tilde{M}_T := \int_0^T \xi_t d\tilde{w}_t/\Gamma_T$. By assumption, $E\tilde{M}_T^2 = \int_0^T E\xi_t^2 dt/\Gamma_T^2 \le c_1/\Gamma_T$. Next, a sequence $\tilde{M}_n$ is constructed that approximates (in a sense) the process $\tilde{M}_T$ on sequences of intervals (in n) so that $\tilde{M}_n \longrightarrow 0$ a.s. and $\sup\limits_{\mathcal{G}(n) \le T < \mathcal{G}(n+1)} |\tilde{M}_T - \tilde{M}_n| \longrightarrow 0$ as $n \longrightarrow \infty$ (where $\mathcal{G}(x)$ is a nondecreasing function).

For $n = 1, 2, \ldots$ consider the sequence $W_n = n^\gamma$ ($\gamma > 1$), and let $\tilde{M}_n := \int_0^{\Gamma^{-1}(W_n)} \xi_t d\tilde{w}_t/W_n$. Then $E\tilde{M}_n^2 = \int_0^{\Gamma^{-1}(W_n)} E\xi_t^2 dt/W_n^2 \le c_1/W_n = c_1/n^\gamma$. Thus, $\Sigma_{n=1}^\infty E\tilde{M}_n^2 < \infty$ and, by the convergence test for random sequences, $\tilde{M}_n \longrightarrow 0$ a.s. as $n \longrightarrow \infty$.

Define the set

$$\tilde{T}_n := \{ T, \Gamma^{-1}(W_n) \le T < \Gamma^{-1}(W_{n+1}) \} \tag{2.1}$$

and the sequence of random variables $Z_n := \sup\limits_{\tilde{T}_n} |\tilde{M}_T - \tilde{M}_n|$. Then

$$Z_n = \sup_{\tilde{T}_n} \left| \int_0^T \xi_t d\tilde{w}_t/\Gamma_T - \int_0^{\Gamma^{-1}(W_n)} \xi_t d\tilde{w}_t/W_n \right| = \sup_{\tilde{T}_n} \left| (1/\Gamma_T - 1/W_n) \int_0^{\Gamma^{-1}(W_n)} \xi_t d\tilde{w}_t + 1/\Gamma_T \int_{\Gamma^{-1}(W_n)}^T \xi_t d\tilde{w}_t \right|$$

$$\le (W_{n+1} - W_n) \left| \int_0^{\Gamma^{-1}(W_n)} \xi_t d\tilde{w}_t \right| / W_n^2 + \sup_{\tilde{T}_n} \left| \int_{\Gamma^{-1}(W_n)}^T \xi_t d\tilde{w}_t \right| / W_n.$$

With the help of the last inequality, Z_n^2 is estimated as

$$Z_n^2 \le 2\frac{(W_{n+1} - W_n)^2}{W_n^4} \left(\int_0^{\Gamma^{-1}(W_n)} \xi_t d\tilde{w}_t \right)^2 + \frac{2}{W_n^2} \left(\sup_{\tilde{T}_n} \left| \int_{\Gamma^{-1}(W_n)}^T \xi_t d\tilde{w}_t \right| \right)^2.$$

The expectation of the second term in this relation is estimated using the inequality from [13], according to which, for a local martingale M_t with $M_0 = 0$ and a quadratic characteristic $\langle M_t \rangle$ for any $p > 0$, there exists a number C_p such that $E(M_t^*)^p \le C_p E([\langle M_t \rangle]^{p/2})$, where $M_t^* = \sup\limits_{s \le t} |M_s|$. Using this result with $p = 2$, we obtain

$$EZ_n^2 \le 2c_1 (W_{n+1} - W_n)^2 / W_n^3 + 2C_p \int_{\Gamma^{-1}(W_n)}^{\Gamma^{-1}(W_{n+1})} E\xi_t^2 dt / W_n^2$$

and, accordingly,

$$EZ_n^2 \le 2c_1 (W_{n+1} - W_n)^2 / W_n^3 + 2c_1 C_p (W_{n+1} - W_n) / W_n^2. \tag{2.2}$$

Consider $(W_{n+1} - W_n)/W_n = ((n + 1)^\gamma - n^\gamma)/n^\gamma = (1 + 1/n)^\gamma - 1$. Let γ_0 be an integer such that $\gamma < \gamma_0$. Obviously, $(1 + 1/n)^\gamma < (1 + 1/n)^{\gamma_0}$. By applying a well-known formula, $(1 + 1/n)^{\gamma_0}$ is expanded in the power series

$$(1 + 1/n)^{\gamma_0} = 1 + \gamma_0/n + \sum_{k=2}^{\gamma_0} C_{\gamma_0}^k / n^k,$$

where

$$C_{\gamma_0}^k = \frac{\gamma_0!}{k!(\gamma_0 - k)!},$$

which yields the estimate $(1 + 1/n)^{\gamma_0} < 1 + c/n$ with a constant c. In what follows, $c > 0$ denotes constants (possibly varying from formula to formula) whose particular values are of no importance. Therefore,

$$(W_{n+1} - W_n)/W_n = (1 + 1/n)^{\gamma} - 1 < c/n. \tag{2.3}$$

Then it obviously follows from (2.2) and (2.3) that $EZ_n^2 < c/n^2$. By the convergence test for random sequences, this estimate implies that $Z_n \longrightarrow 0$ a.s. as $n \longrightarrow \infty$, whence $\tilde{M}_T = \int_0^T \xi_t d\tilde{w}_t/\Gamma_T \longrightarrow 0$ a.s. as $T \longrightarrow \infty$. The lemma is proved.

Lemma 2. *Let* $\{Y_t\}_{t=0}^{\infty}$ *be a stochastic process such that* $EY_t = 0$ *for any* $t \geq 0$, *and let* Γ_t *be a nondecreasing function with* $\Gamma_0 = 0$. *If there exists a number* $A > 0$ *such that, for any* a *and* b : $A \leq a < b$, *we have* $\int_0^b\int_0^b E(Y_tY_s)dsdt \leq c_{21}\Gamma_b$, $\int_a^b \sqrt{EY_t^2}dt \leq c_{22}(\Gamma_b - \Gamma_a)$ (*where* $c_{21}, c_{22} > 0$ *are constants*) *and* $\lim_{t\to\infty}\Gamma_t = \infty$, *then* $\int_0^T Y_t dt/\Gamma_T \longrightarrow 0$ a.s. *as* $T \longrightarrow \infty$.

Proof. Define the process $\tilde{Y}_T := \int_0^T Y_t dt/\Gamma_T$. According to a well-known relation for mean-square integrable processes (see [12]), we have $E\tilde{Y}_T^2 = E(\int_0^T Y_t^2 dt)/\Gamma_T^2 = \int_0^T\int_0^T E(Y_tY_s)dsdt/\Gamma_T^2$. Moreover, by assumption, $\int_0^T\int_0^T E(Y_tY_s)dsdt/\Gamma_T^2 \leq c_{21}/\Gamma_T$. As in the proof of Lemma 1, a sequence $\tilde{Y}_n$ approximating the process $\tilde{Y}_T$ is constructed by setting $W_n = n^{\gamma}$, $(\gamma > 1)$. More precisely, let $\tilde{Y}_n := \int_0^{\Gamma^{-1}(W_n)} Y_t dt/W_n$ $(n = 1, 2, \ldots)$. We find that

$$E\tilde{Y}_n^2 = E\left(\int_0^{\Gamma^{-1}(W_n)} Y_t dt\right)^2 / W_n^2 \leq c_{21}\Gamma(\Gamma^{-1}(W_n))/W_n^2 = c_{21}/n^{\gamma}.$$

Therefore, $\Sigma_{n=1}^{\infty} E\tilde{Y}_n^2 < \infty$ and, by the convergence test for random sequences, $\tilde{Y}_n \longrightarrow 0$ a.s. as $n \longrightarrow \infty$. For the set $\tilde{T}_n$ defined in (2.1), consider the sequence

$$Z_n := \sup_{\tilde{T}_n}|\tilde{Y}_T - \tilde{Y}_n| = \sup_{\tilde{T}_n}|Y_T/\Gamma_T - Y_n/W_n| = \sup_{\tilde{T}_n}\left|\int_0^T Y_t dt/\Gamma_T - \int_0^{\Gamma^{-1}(W_n)} Y_t dt/W_n\right|$$

$$= \sup_{\tilde{T}_n}\left|(1/\Gamma_T - 1/W_n)\int_0^{\Gamma^{-1}(W_n)} Y_t dt + 1/\Gamma_T \int_{\Gamma^{-1}(W_n)}^{T} Y_t dt\right|$$

$$\leq (W_{n+1} - W_n)\int_0^{\Gamma^{-1}(W_n)} |Y_t| dt/W_n^2 + \int_{\Gamma^{-1}(W_n)}^{\Gamma^{-1}(W_{n+1})} |Y_t| dt/W_n.$$

Then

$$Z_n^2 \leq \frac{2(W_{n+1} - W_n)^2}{W_n^4}\int_0^{\Gamma^{-1}(W_n)}\int_0^{\Gamma^{-1}(W_n)} |Y_tY_s| dsdt + \frac{2}{W_n^2}\int_{\Gamma^{-1}(W_n)}^{\Gamma^{-1}(W_{n+1})}\int_{\Gamma^{-1}(W_n)}^{\Gamma^{-1}(W_{n+1})} |Y_tY_s| dsdt.$$

Using the Cauchy–Schwarz inequality $E|Y_tY_s| \le \sqrt{EY_t^2}\sqrt{EY_s^2}$, the condition of the lemma, and (2.3), we obtain

$$EZ_n^2 \le 2(W_{n+1}-W_n)^2\left(\int_0^{\Gamma^{-1}(W_n)}\sqrt{EY_t^2}dt\right)^2/W_n^4 + 2\left(\int_{\Gamma^{-1}(W_n)}^{\Gamma^{-1}(W_{n+1})}\sqrt{EY_t^2}dt\right)^2/W_n^2$$

$$\le 2c_{21}^2(W_{n+1}-W_n)^2W_n^2/W_n^4 + 2c_{22}^2(W_{n+1}-W_n)^2/W_n^2 < c/n^2.$$

Therefore, $Z_n \longrightarrow 0$ a.s. as $n \longrightarrow \infty$, and $\tilde{Y}_T = \int_0^T Y_t dt/\Gamma_T \longrightarrow 0$ a.s. as $T \longrightarrow \infty$. The lemma is proved.

Lemma 3. *Let a one-dimensional stochastic process* $\{z_t\}_{t=0}^{\infty}$ *be described by the equation*

$$dz_t = -z_t dt + \alpha_t' dw_t, \quad z_0 = 0, \tag{2.4}$$

where α_t *is a d-dimensional bounded vector function of time t and* w_t *is a d-dimensional Wiener process* (*here*, ' *denotes transposition*). *Let* $\int_0^T\|\alpha_t\|^2dt > 0$, *at least, for sufficiently large T. Then*, *as* $T \longrightarrow \infty$,

$$\frac{z_T^2}{\int_0^T\|\alpha_t\|^2dt} \longrightarrow 0 \text{ a.s.} \tag{2.5}$$

Proof. By using Ito's formula, we write the equation for the dynamics of z_t^2,

$$dz_t^2 = -2z_t^2dt + \|\alpha_t\|^2dt + 2\alpha_t'z_tdw_t, \quad z_0^2 = 0, \tag{2.6}$$

and the corresponding equation for Ez_t^2:

$$d(Ez_t^2) = -2Ez_t^2dt + \|\alpha_t\|^2dt, \quad Ez_0^2 = 0. \tag{2.7}$$

Subtracting Eq. (2.7) from (2.6) yields

$$d(z_t^2 - Ez_t^2) = -2(z_t^2 - Ez_t^2)dt + 2\alpha_t'z_tdw_t, \quad z_0^2 - E(z_0^2) = 0. \tag{2.8}$$

In integral form, (2.8) is written as

$$z_T^2 - E(z_T^2) = -2\int_0^T(z_t^2 - E(z_t^2))dt + 2\int_0^T\alpha_t'z_tdw_t. \tag{2.9}$$

Dividing both sides of (2.9) by $\int_0^T\|\alpha_t\|^2dt$ yields

$$\frac{(z_T^2 - Ez_T^2)}{\int_0^T\|\alpha_t\|^2dt} = -\frac{2\int_0^T(z_t^2 - Ez_t^2)dt}{\int_0^T\|\alpha_t\|^2dt} + \frac{2\int_0^T\alpha_t'z_tdw_t}{\int_0^T\|\alpha_t\|^2dt}. \tag{2.10}$$

Now the task is to analyze the asymptotic behavior of the terms on the right-hand side of (2.10). The following two cases are considered separately: (i) $\int_0^T\|\alpha_t\|^2dt \longrightarrow \infty$, $T \longrightarrow \infty$ and (ii) $\int_0^T\|\alpha_t\|^2dt < \infty$.

Consider case (i) first. Let $\Gamma_T := \int_0^T \|\alpha_t\|^2 dt$, and let α_t^j and w_t^j be the jth components of the vector function α_t and the Wiener process w_t, respectively. Then $\int_0^T \alpha_t' z_t dw_t = \sum_{j=1}^d \int_0^T \alpha_t^j z_t dw_t^j$. Let us verify that the conditions of Lemma 1 are satisfied for each of the terms of the last sum.

Note that z_t is a Gaussian stochastic process with the expectation $Ez_t = 0$ and the covariance function

$$E(z_t, z_s) = e^{-|t-s|} Ez_\tau^2 = e^{-|t-s|} \int_0^\tau e^{-2(\tau-\theta)} \|\alpha_\theta\|^2 d\theta, \quad \tau = \min(t, s). \tag{2.11}$$

The bounded function $Ez_t^2 = e^{-2t} \int_0^t e^{2s} \|\alpha_s\|^2 ds$ satisfies Eq. (2.7), whence

$$\int_0^t Ez_s^2 ds = 1/2 \int_0^t \|\alpha_s\|^2 ds - Ez_t^2/2 \le 1/2 \int_0^t \|\alpha_s\|^2 ds = 1/2\Gamma_t. \tag{2.12}$$

Therefore, $\int_0^T (\alpha_t^j)^2 Ez_t^2 dt \le c\Gamma_T$ for any $j = 1, 2, \ldots, d$, and, by Lemma, 1 $\int_0^T \alpha_t' z_t dw_t / \int_0^T \|\alpha_t\|^2 dt \longrightarrow 0$ a.s. as $T \longrightarrow \infty$.

To analyze the asymptotic behavior of the first term in (2.10), we need Lemma 2 for $Y_t = z_t^2 - Ez_t^2$. First, we show that

$$E(Y_t Y_s) = 2(E(z_t, z_s))^2 = 2e^{-2|t-s|}(Ez^2_{\min(t,s)})^2. \tag{2.13}$$

For $t \ge s$, the solution of Eq. (2.4) is written as

$$z_t = e^{-t} \int_0^t e^\tau \alpha_\tau' dw_\tau = e^{-(t-s)} z_s + e^{-t} \int_s^t e^\tau \alpha_\tau' dw_\tau.$$

Then $z_t z_s = e^{-(t-s)} z_s^2 + e^{-t} z_s \int_s^t e^\tau \alpha_\tau' dw_\tau$. Note that z_s and $\int_s^t e^\tau \alpha_\tau' dw_\tau$ are independent random variables (for $s \le t$). Accordingly, $E(z_t^2 z_s^2) = e^{-2(t-s)} Ez_s^4 + e^{-2t} Ez_s^2 \int_s^t e^{2\tau} \|\alpha_\tau\|^2 dt$. By using the expression obtained previously for Ez_t^2, it is easy to show that

$$e^{-2t} \int_s^t e^{2\tau} \|\alpha_\tau\|^2 dt Ez_s^2 = Ez_s^2 Ez_t^2 - e^{-2(t-s)} (Ez_s^2)^2.$$

Then

$$E(Y_t Y_s) = E(z_t^2 z_s^2) - Ez_s^2 Ez_t^2 = e^{-2(t-s)} (Ez_s^4 - (Ez_s^2)^2).$$

From (2.7) and (2.8), we derive the equations

$$d(Ez_t^4 - (Ez_t^2)^2) = -4(Ez_t^4 - (Ez_t^2)^2)dt + 4\|\alpha_t^2\|dt,$$

$$d((Ez_t^2)^2) = -4(Ez_t^2)^2 dt + 2\|\alpha_t^2\|dt.$$

Note that $Ez_s^4 - (Ez_s^2)^2 = 2(Ez_s^2)^2$; i.e., $E(Y_t Y_s) = 2e^{-2(t-s)}(Ez_s^2)^2$ for $s \le t$. The case $s > t$ is treated in a similar manner. Now, we check the conditions of Lemma 2. From (2.11) and (2.13), it follows that

$$\int_0^b \int_0^b E(Y_t Y_s) ds dt \le c \int_0^b \int_0^b E(z_t, z_s) ds dt = c \int_0^b e^{-t} \int_0^t e^s Ez_s^2 ds dt + c \int_0^b e^{-s} \int_t^b e^t Ez_t^2 ds dt. \tag{2.14}$$

Define $S_t := e^{-t}\int_0^t e^s Ez_s^2$. The first term in (2.14) represents $\int_0^b S_t dt$, where the function S_t satisfies the equation $dS_t = -S_t dt + Ez_t^2 dt$, $S_0 = 0$. Therefore, $\int_0^b S_t dt = \int_0^b Ez_t^2 dt - S_b$. Concerning the second term, we note that $c\int_0^b e^{-s}\int_t^b e^t Ez_t^2 ds dt = \int_0^b (e^{-t} - e^{-b})e^t Ez_t^2 dt = \int_0^b Ez_t^2 dt - S_b$. By applying (2.12), relation (2.14) is transformed into

$$\int_0^b\int_0^b E(Y_t Y_s) ds dt \le c\Gamma_b. \tag{2.15}$$

The second condition in Lemma 2 holds as well. Indeed, by virtue of (2.13),

$$\int_a^b \sqrt{EY_t^2} dt = \sqrt{2}\int_a^b Ez_t^2 dt. \tag{2.16}$$

Therefore,

$$\int_a^b \sqrt{EY_t^2} dt = \sqrt{2}\int_a^b \|\alpha_t\|^2 dt - \sqrt{2}Ez_b^2 + \sqrt{2}Ez_a^2 \le c(\Gamma_b - \Gamma_a) \quad \text{for} \quad a > A.$$

By Lemma 2,

$$\int_0^T Y_t dt \Big/ \int_0^T \|\alpha_t\|^2 dt = \int_0^T (z_t^2 - Ez_t^2) dt \Big/ \int_0^T \|\alpha_t\|^2 dt \longrightarrow 0 \quad \text{a.s. as} \quad T \longrightarrow \infty.$$

Taking into account that Ez_t^2 is bounded and using the above relations and (2.10), we complete the proof of (2.5) in case (i).

Consider case (ii). Let us show that $z_T \longrightarrow 0$ a.s. as $T \longrightarrow \infty$. For this purpose, we use representation (2.9). Define $\Theta_T := \int_0^T (z_t^2 - Ez_t^2) dt = \int_0^T Y_t dt$. It is well known (see [12]) that the limit $\Theta_\infty = \lim_{T\to\infty} \Theta_T$ exists if so does the improper integral $\int_0^\infty \sqrt{EY_t^2} dt$. The validity of the latter can be shown using (2.12) and (2.16).

It remains to analyze the asymptotic behavior of the martingale $M_T := \int_0^T \alpha_t' z_t dw_t$. Making a random change of time, we have $M_T = \bar{w}_{\langle M_T\rangle}$, where $\{\bar{w}_t\}_{t=0}^\infty$ is a Wiener process ans $\langle M_T\rangle = \int_0^T \|\alpha_t\|^2 z_t^2 dt$. By analogy with the proof of the existence of Θ_∞, it is easy to see that $\langle M_T\rangle \longrightarrow \langle M_\infty\rangle$, $T \longrightarrow \infty$. Therefore, $M_\infty = \bar{w}_{\langle M_\infty\rangle}$ exists as well. It can be shown (see [8]) that $Ez_T^2 \longrightarrow 0$, $T \longrightarrow \infty$. Using the above results concerning the convergence of the terms on the right-hand side of (2.9), we obtain $z_T \longrightarrow z_\infty$ a.s. as $T \longrightarrow \infty$. Since $Ez_T^2 \longrightarrow 0$, $T \longrightarrow \infty$, we conclude that, by definition, $z_T \longrightarrow \tilde{z}_\infty$ in mean square and, moreover, $\tilde{z}_\infty = 0$. As was noted in [12], if there are two types of convergence for a random sequence, then the limiting random variables a.s. coincide; i.e., $z_\infty = \tilde{z}_\infty = 0$. Lemma 3 is proved.

Proof of Theorem 1. Consider the process $\tilde{X}_t$ defined by the equation

$$d\tilde{X}_t = -I\tilde{X}_t dt + G_t dw_t, \quad \tilde{X}_0 = \bar{0}, \tag{2.17}$$

where $\bar{0}$ denotes a zero vector. For the ith component of this vector process, we can write $d\tilde{X}_{it} = -\tilde{X}_{it}dt + G_t^i dw_t$ and $\tilde{X}_{i0} = 0$, where G_t^i is the ith row of the matrix G_t. Obviously, this equation implies that $\tilde{X}_{it}$ corresponds to the process in Lemma 3 with $z_t = \tilde{X}_{it}$ and $\alpha_t = (G_t^i)'$. If $\int_0^\infty \|(G_t^i)\|^2 dt > 0$, then the lemma holds; otherwise, it is clear that $\tilde{X}_{it}^2 \longrightarrow 0$, $T \longrightarrow \infty$. Thus, as $T \longrightarrow \infty$,

$$\|\tilde{X}_T\|^2 / \int_0^T \|G_t\|^2 dt \longrightarrow 0 \text{ a.s.} \tag{2.18}$$

Define the process $Z_t := X_t - \tilde{X}_t$. By virtue of (1.1) and (2.17), it is described by the equation $dZ_t = A_t Z_t dt + (A_t + I)\tilde{X}_t dt$, $Z_0 = x$, whose solution is given by $Z_T = \Phi_{\mathcal{A}}(T, 0)x + \int_0^T \Phi_{\mathcal{A}}(T, t)(A_t + I)\tilde{X}_t dt$, where $\Phi_{\mathcal{A}}(t, s)$ is a fundamental matrix for the matrix function $\mathcal{A}_t := A_t$. Dividing both sides by $(\Gamma_T)^{1/2} = (\int_0^T \|G_t\|^2 dt)^{1/2}$ and using Assumption 1, since the matrix function A_t is bounded, we obtain the estimate

$$\|Z_T\|(\Gamma_T)^{-1/2} \le \kappa_1 e^{-\kappa_2 T}(\Gamma_T)^{-1/2} x + c e^{-\kappa_1 T}\int_0^T e^{\kappa_1 t}\|\tilde{X}_t\|(\Gamma_t)^{-1/2} dt. \tag{2.19}$$

Applying property (2.18), we find that, for any number $\epsilon > 0$, there exists a.s. a finite time $t_0(\omega)$ such that $\|\tilde{X}_t\|(\Gamma_t)^{-1/2} < \epsilon$ for any $t > t_0$ and almost all ω. Substituting this relation into (2.19) yields

$$\|Z_T\|(\Gamma_T)^{-1/2} \le \kappa_1 e^{-\kappa_2 T}(\Gamma_T)^{-1/2} x + \int_0^{t_0} c e^{-\kappa_1(T-t)}\|\tilde{X}_t\|(\Gamma_t)^{-1/2} dt + c\epsilon \int_{t_0}^T e^{-\kappa_1(T-t)} dt. \tag{2.20}$$

Since the second term in (2.20) tends a.s. to zero as $T \longrightarrow \infty$, in a similar manner, for arbitrary $\epsilon_1 > 0$, we obtain $\int_0^{t_0} c e^{-\kappa_1(T-t)}\|\tilde{X}_t\|\Gamma_t^{-1/2} dt < \epsilon_1$ a.s. for any $T > t_1(\omega)$. Accordingly,

$$\|Z_T\|(\Gamma_T)^{-1/2} \le \kappa_1 e^{-\kappa_2 T}(\Gamma_T)^{-1/2} x + \epsilon_1 + c\epsilon \int_{t_0}^T e^{-\kappa_1(T-t)} dt < \epsilon_2 \quad \text{a.s. for any} \quad T > t_2(\omega),$$

where $t_1(\omega)$ and $t_2(\omega)$ are a.s. finite.

Thus, $\|Z_T\|(\Gamma_T)^{-1/2} \longrightarrow 0$ a.s. as $T \longrightarrow \infty$. By the definition of Z_t, it is obvious that $\|X_T\|^2 / \int_0^T \|G_t\|^2 dt \longrightarrow 0$ a.s. as $T \longrightarrow \infty$ as well, which completes the proof of the theorem.

2.2. Modeling Example

Consider the equation

$$dX_t = -X_t dt + (1 - 1/\ln(t+3))^{1/2}/\ln^{1/4}(t+3) dw_t, \quad X_0 = 1. \tag{2.21}$$

It is easy to see that the function $\Phi_{\mathcal{A}}(t, s) = \exp(s - t)$ constructed for $\mathcal{A}_t = -1$ satisfies Assumption 1 and that the variance of integral noise actions

$$\Gamma_T = \int_0^T \|G_t\|^2 dt = \int_0^T \frac{1 - 1/\ln(t+3)}{\ln^{1/2}(t+3)} dt = \frac{(T+3)}{\sqrt{\ln(T+3)}} - \frac{3}{\sqrt{\ln 3}}$$

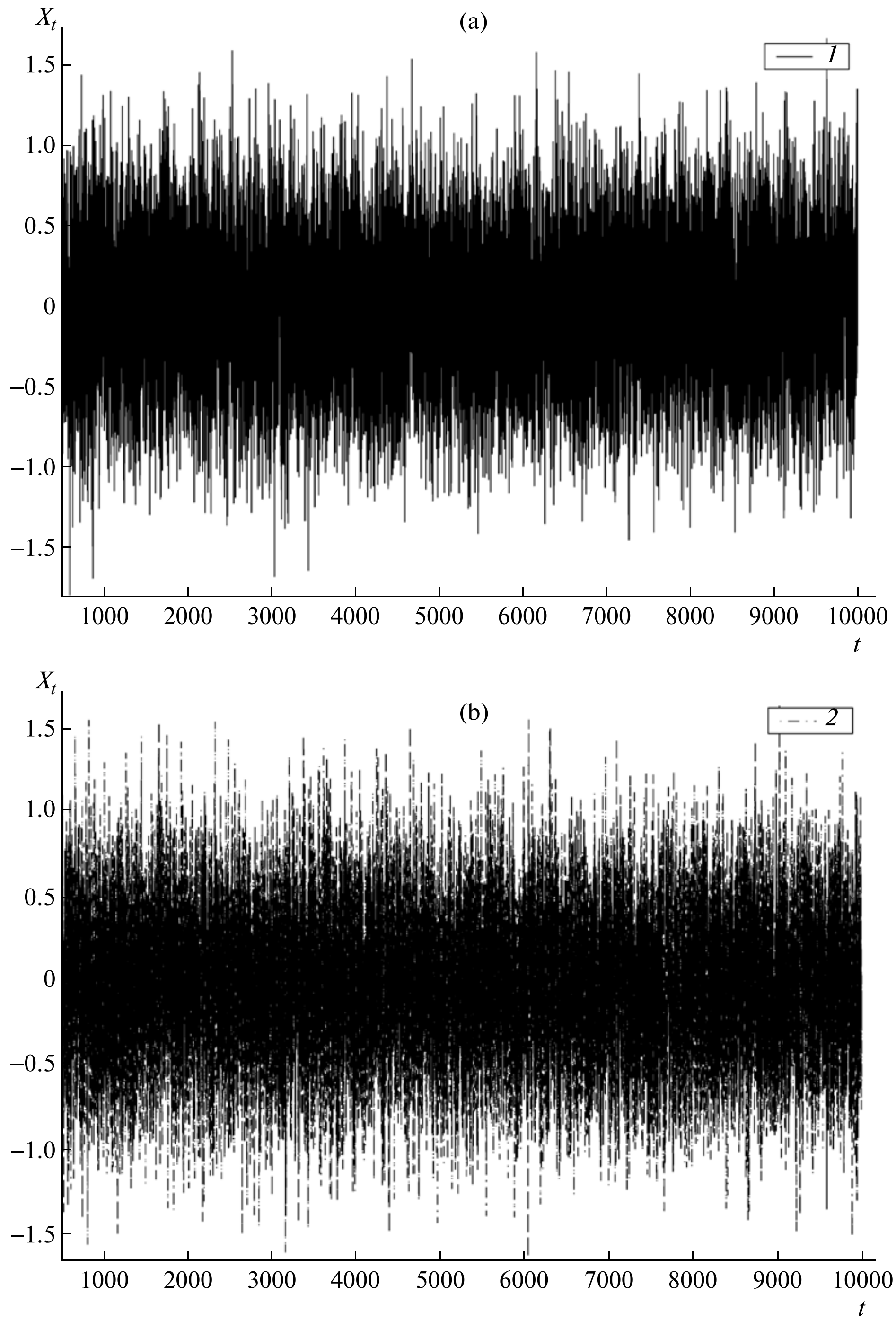

Fig. 1.

increases indefinitely as $T \longrightarrow \infty$. Figures 1a and 1b show the trajectories of $X_t(\omega)$ for two different ω for large t. It was shown in [5] that, for $n = 1$ (see Eq. (1.1)), the relation $G_T^2 \ln T \longrightarrow \infty$, $T \longrightarrow \infty$, is sufficient for $\limsup_{T \to \infty} |X_T| = +\infty$. Obviously, this condition holds for G_t corresponding to the process given by (2.21) and the normalization with the help of $\Gamma_T = (T+3)/\sqrt{\ln(T+3)} - 3/\sqrt{\ln 3}$ is required for the process $(X_T^N)^2 = X_T^2/\Gamma_T$ to tend asymptotically a.s. to zero, which can be seen in Figs. 2a and 2b for two realizations of the process for $T = 10000$.

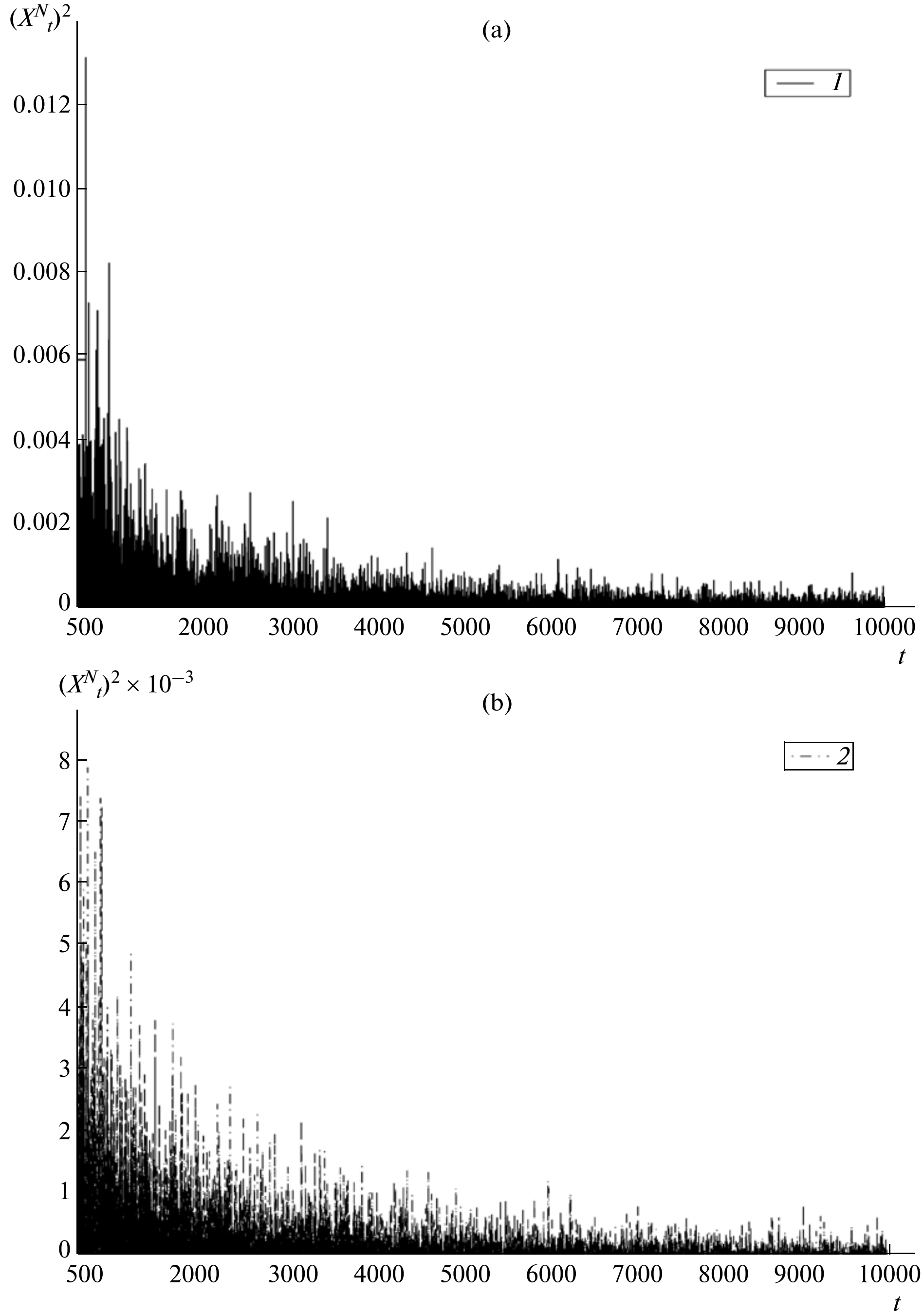

Fig. 2.

3. STOCHASTIC OPTIMALITY FOR A LINEAR REGULATOR

3.1. Classical Linear-Quadratic Regulator

Now consider an n-dimensional controlled stochastic process V_t, $t \geq 0$, described by the equation

$$dV_t = C_t V_t dt + B_t V_t dt + G_t dw_t, \quad V_0 = x, \tag{3.1}$$

where the initial state x is nonrandom; w_t $(t \geq 0)$ is a d-dimensional standard Wiener process; U_t $(t \geq 0)$ is an admissible control or a k-dimensional stochastic process adapted to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $\mathcal{F}_t =$

$\sigma\{w_s, s \le t\}$ such that Eq. (3.1) is solvable; and C_t, B_t, and G_t, $t \ge 0$, are bounded matrix functions of time of suitable sizes such that (3.1) makes sense.

The set of admissible controls is denoted by $\mathcal{U}$. For every $T > 0$, the objective functional is defined as the random variable

$$J_T(U^T) = \int_0^T (V_t' Q_t V_t + U_t' R_t U_t) dt, \tag{3.2}$$

where $U^T = \{U_t\}_{t \le T}$ is the restriction of the control $U \in \mathcal{U}$ to the interval $[0, T]$ and Q_t and R_t $(t \ge 0)$ are positive semidefinite and positive definite bounded matrix functions of time, respectively. Moreover, $U^T \in \mathcal{U}^T$, where $\mathcal{U}^T$ is the set of admissible controls considered on $[0, T]$. The parameters of the control system satisfy the following assumptions.

Assumption 2. The functions C_t, B_t, Q_t, and R_t, $t \ge 0$, are such that there exists an absolutely continuous bounded function Π_t $(t \ge 0)$ with values in the set of positive semidefinite symmetric matrices that satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t C_t + C_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0 \tag{3.3}$$

and such that the fundamental matrix $\Phi_{\mathcal{A}}(t, s)$ for the function $\mathcal{A}_t := C_t - B_t R_t^{-1} B_t' \Pi_t$ satisfies exponential estimate (1.2).

Assumption 3. There is a constant $c_0 > 0$ such that any pair $(x_t, u_t)_{t \le T}$ obeying the equation

$$dx_t = C_t x_t dt + B_t u_t dt, \quad x_0 = 0, \tag{3.4}$$

satisfies the inequality

$$\|x_T\|^2 + \int_0^T \|x_t\|^2 dt \le c_0 \int_0^T (x_t' Q_t x_t + u_t' R_t u_t) dt. \tag{3.5}$$

The standard conditions on the parameters of system (3.1), (3.2), which are sufficient for Assumptions 2 and 3 to hold, can be found in [6] (see also [5]).

Definition 1. A control $U^* \in \mathcal{U}$ is said to be *almost surely optimal on an infinite horizon* or *optimal in the sense of the generalized stochastic long-run average* if it solves the problem

$$\limsup_{T \to \infty} \frac{J_T(U)}{\int_0^T \|G_t\|^2 dt} \longrightarrow \inf_{U \in \mathcal{U}} \text{ with probability 1.} \tag{3.6}$$

This definition generalizes the well-known concept of the stochastic long-run average (see, e.g., [14]). Under Assumptions 2 and 3, the control U^* is defined as

$$U_t^* = -R_t^{-1} B_t' \Pi_t V_t^*, \tag{3.7}$$

where the process $\{V_t^*\}_{t=0}^{\infty}$ is defined by the equation

$$dV_t^* = (C_t - B_t R_t^{-1} B_t' \Pi_t) V_t^* dt + G_t dw_t, \quad V_0^* = x. \tag{3.8}$$

In [8] the stochastic optimality of U^* in the sense of the solution to problem (3.6) was proved under some additional assumptions, for example, $\|G_t\|^2 \ln t \longrightarrow \infty$, $t \longrightarrow \infty$. It will be shown later that these constraints can be avoided by applying the results of Section 1.

Theorem 2. *Let Assumptions* 2 *and* 3 *hold, and let* $\int_0^T \|G_t\|^2 dt \longrightarrow \infty$ *as* $T \longrightarrow \infty$. *Then the control* U_t^* *defined by* (3.7) *and* (3.8) *solves problem* (3.6).

Proof. Let $U \in \mathcal{U}$ be an arbitrary fixed admissible control. By using Eq. (3.1), we find the corresponding process V_t and define $x_t := V_t - V_t^*$ and $u_t = U_t - U_t^*$. Note that the pair (x_t, u_t) satisfies Eq. (3.4). Under Assumptions 1 and 2, the following estimate was obtained in [6]:

$$J_T(U^*) \le J_T(U) + \tilde{c}_1 \|V_T^*\|^2 + \mathcal{R}_T, \tag{3.9}$$

where $\mathcal{R}_T := -\tilde{c}_2 \int_0^T \|G_t'\Pi_t x_t\|^2 dt - 2\int_0^T x_t'\Pi_t G_t dw_t$ and $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ are constants. Normalizing (3.9) by the multiplier $\Gamma_T = \int_0^T \|G_t\|^2 dt$ yields

$$J_T(U^*)(\Gamma_T)^{-1} \le J_T(U)(\Gamma_T)^{-1} + \tilde{c}_1 \|V_T^*\|^2 (\Gamma_T)^{-1} + \mathcal{R}_T(\Gamma_T)^{-1}. \tag{3.10}$$

According to [6], $\limsup_{T\to\infty} \mathcal{R}_T g_T \le 0$ for any function $g_T > 0$ such that $g_T \longrightarrow 0$, $T \longrightarrow \infty$. By assumption, we can set $g_T = (\Gamma_T)^{-1}$. The process V_t^* satisfies Assumption 1. Consequently, Theorem 1 holds for it. In view of the obtained relations, passage to the limit as $T \longrightarrow \infty$ in (3.10) produces

$$\limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T \|G_t\|^2 dt} \le \limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T \|G_t\|^2 dt}.$$

To complete the proof of the theorem, we show that the criterion $\limsup_{T\to\infty}(J_T(U^*)/\int_0^T \|G_t\|^2 dt)$ has a finite value. Combining (3.3) and (3.8) with Ito's formula yields

$$d((V_t^*)'\Pi_t V_t^*) = -\{(V_t^*)'Q_t V_t^* + (U_t^*)'R_t U_t^*\}dt + tr(G_t'\Pi_t G_t)dt + 2(V_t^*)'\Pi_t G_t dw_t;$$

i.e.,

$$J_T(U^*) = x'\Pi_0 x - (V_T^*)'\Pi_T V_T^* + \int_0^T tr(G_t'\Pi_t G_t)dt + 2\int_0^T (V_t^*)'\Pi_t G_t dw_t. \tag{3.11}$$

Let us verify that the conditions of Lemma 1 hold for the last term in (3.11). The solution of Eq. (3.8) has the form $V_t^* = \Phi_{\mathcal{A}}(t,0)x + \int_0^t \Phi_{\mathcal{A}}(t,s)G_s dw_s$, where $\Phi_{\mathcal{A}}(t,s)$ is the fundamental matrix for the function $\mathcal{A}_t := C^t - B_t B_t'\Pi_t$. By the Ito isometry, it follows from exponential estimate (1.2) that $E\|V_t^*\|^2 \le ce^{-2\kappa_2 t}\|x\|^2 + c\int_0^t e^{-2\kappa_2(t-s)}\|G_s\|^2 ds$. Since G_t is bounded, this relation implies that $E\|V_t^*\|^2$ is bounded as well. The boundedness of Π_t implies that, for each term $\xi_t^l = \int_0^T ((V_t^*)'\Pi_t G_t)^l dw_t^l$, $(l = 1, \ldots, d)$ in the sum $\int_0^T (V_t^*)'\Pi_t G_t dw_t$, we have $\int_a^b E(\xi_t^l)^2 dt \le c\int_a^b \|G_t\|^2 dt$. By Lemma 1, $(\Gamma_T)^{-1}\int_0^T (V_t^*)'\Pi_t G_t dw_t \longrightarrow 0$ a.s. as $T \longrightarrow \infty$. Then the normalized value (3.11) is asymptotically estimated as $\limsup_{T\to\infty}(J_T(U^*)/\int_0^T \|G_t\|^2 dt) \le \limsup_{T\to\infty}(\int_0^T tr(G_t'\Pi_t G_t)dt/\int_0^T \|G_t\|^2 dt) \le c_J$, where $c_J > 0$ is a constant. The theorem is proved.

3.2. Linear-Quadratic Regulator with Discounting

Consider an n-dimensional controlled stochastic process $\tilde{X}_t$, $t \ge 0$, described by the equation

$$d\tilde{X}_t = C\tilde{X}_t dt + B\tilde{U}_t dt + G dw_t, \quad \tilde{X}_0 = \tilde{x}, \tag{3.12}$$

where $\tilde{U}$ is an admissible control in the sense described in Section 3.1; C, B, and G are matrices of suitable sizes; and $\tilde{x}$ is a nonrandom vector. The set of admissible controls is denoted by $\tilde{\mathcal{U}}$. Assume that $\|G\| > 0$.

Let $\tilde{x}_0$ and $\tilde{u}_0$ be a fixed state vector and a fixed control vector such that $C\tilde{x}_0 + B\tilde{u}_0 = \bar{0}$. The cost functional taking into account the losses due to the deviations of $\tilde{X}_t$ from $\tilde{x}_0$ and $\tilde{U}_t$ from $\tilde{u}_0$ on [0, T] is defined as

$$J_T(\tilde{U}^T) = \int_0^T f_t\{(\tilde{X}_t' - \tilde{x}_0')Q(\tilde{X}_t - \tilde{x}_0) + (\tilde{U}_t' - \tilde{u}_0')R(\tilde{U}_t - \tilde{u}_0)\}dt, \tag{3.13}$$

where $\tilde{U}^T = \{\tilde{U}_t\}_{t \le T}$ is the restriction of the control $\tilde{U} \in \tilde{\mathcal{U}}$ to the interval [0, T] and Q and R are positive semidefinite and positive definite symmetric matrices, respectively. Moreover, $\tilde{U}^T \in \tilde{\mathcal{U}}^T$, where $\tilde{\mathcal{U}}^T$ is the set of admissible controls considered on [0, T], and f_t is a discount function having the properties described in the following assumption.

Assumption 4. The discount function $f_t > 0$, $t \ge 0$, is nonincreasing and differentiable; $f_0 = 1$; $f_t \longrightarrow 0$, $t \longrightarrow \infty$; and $\phi_t = -\dot{f}_t/f_t$ is bounded.

For example, the function $f_t = e^{-\gamma t}$, $\gamma > 0$, corresponds to traditional exponential discounting, while $f_t = 1/(1 + \theta t)^{\theta_1/\theta}$, $\theta_1, \theta > 0$, to hyperbolic discounting.

Let us define almost sure optimality for a system with discounting. For this purpose, the functional is normalized using the accumulated discount approach proposed in [15], where the average optimality of controlled stochastic processes on an infinite horizon with a cost functional involving a discount function was studied.

Definition 2. A control $\tilde{U}^* \in \tilde{\mathcal{U}}$ is said to be *almost surely optimal on an infinite time horizon in the problem with discounting* if it solves the problem

$$\limsup_{T \to \infty} \frac{J_T(\tilde{U})}{\int_0^T f_t dt} \longrightarrow \inf_{\tilde{U} \in \tilde{\mathcal{U}}} \quad \text{with probability 1.} \tag{3.14}$$

As was noted in [8], the parameters of system (3.12), (3.13) do not satisfy a property from the set of sufficient conditions allowing one to consider control problems for a linear-quadratic regulator when $T \longrightarrow \infty$. More precisely, the matrix function f_tR in (3.13) has to be bounded away from zero, which is obviously not satisfied if Assumption 4 holds for f_t. However, it was shown in [8] that, in the case $n = 1$, system (3.12), (3.13) can be reduced to standard regulator (3.1), (3.2) by changing variables. Define

$$X_t := \sqrt{f_t}(\tilde{X}_t - \tilde{x}_0), \quad U_t := \sqrt{f_t}(\tilde{U}_t - \tilde{u}_0). \tag{3.15}$$

The dynamics of the process X_t, $t \ge 0$, is described by the equation

$$dX_t = (C - 1/2\phi_t I)X_t dt + BU_t dt + \sqrt{f_t}G dw_t, \quad X_0 = \tilde{x} - \tilde{x}_0. \tag{3.16}$$

In the new notation, functional (3.13) becomes

$$J_T(U) = \int_0^T (X_t'QX_t + U_t'RU_t)dt. \tag{3.17}$$

Obviously, $J_T(\tilde{U}) = J_T(U)$. Let

$$C_t = C - 1/2\phi_t I, \quad B_t = B, \quad G_t = \sqrt{f_t}G, \quad Q_t = Q, \quad R_t = R, \quad x = \tilde{x} - \tilde{x}_0,$$

and the set $\mathcal{U}$ is specified by $\tilde{\mathcal{U}}$. Then system (3.16), (3.17) corresponds to the form of (3.1), (3.2). Moreover, the optimality criterion from Definition 1 then becomes

$$\limsup_{T \to \infty} \frac{J_T(U)}{\|G\|^2 \int_0^T f_t dt} \longrightarrow \inf_{U \in \mathcal{U}} \quad \text{with probability 1,}$$

which coincides with (3.14) in Definition 2. Therefore, if the parameters of system (3.16), (3.17) satisfy the conditions of Theorem 2, then the control $\tilde{U}_t^* = U_t^*/\sqrt{f_t} + \tilde{u}_0$, where U^* is given by (3.7), is almost surely optimal on an infinite horizon in the problem with discounting. This result can be stated as the following theorem.

Theorem 3. *Let Assumption* 4 *hold and* $\int_0^T f_t dt \longrightarrow \infty$ *as* $T \longrightarrow \infty$. *Additionally, assume that the parameters of system* (3.12), (3.13) *are such that Assumptions* 2 *and* 3 *hold for* $C_t = C - 1/2\phi_t I$, $B_t \equiv B$, $Q_t \equiv Q$, *and* $R_t \equiv R$. *Then the control*

$$\tilde{U}_t^* = -R^{-1}B'\Pi_t(\tilde{X}_t^* - \tilde{x}_0) + \tilde{u}_0, \tag{3.18}$$

where the process $\{\tilde{X}_t^*\}_{t=0}^{\infty}$ *is defined by the equation*

$$d\tilde{X}_t^* = (C - BR^{-1}B'\Pi_t)(\tilde{X}_t^* - \tilde{x}_0)dt + Gdw_t, \quad \tilde{X}_0^* = \tilde{x}, \tag{3.19}$$

is a solution of problem (3.14).

Proof. By assumption, Theorem 1 holds for control system (3.16), (3.17). Using the inverse of transformation (3.15), i.e., $\tilde{X}_t^* = X_t^*/\sqrt{f_t} + \tilde{x}_0$, $\tilde{U}_t^* = U_t^*/\sqrt{f_t} + \tilde{x}_0$, we obtain $\tilde{U}_t^* = -R^{-1}B'\Pi_t(\tilde{X}_t^* - \tilde{x}_0) + \tilde{u}_0$ and derive the equation

$$d\tilde{X}_t^* = C\tilde{X}_t^*dt - BR^{-1}B'\Pi_t(\tilde{X}_t^* - \tilde{x}_0)dt + B\tilde{u}_0dt + Gdw_t, \quad \tilde{X}_0^* = \tilde{x}.$$

By assumption, $C\tilde{x}_0 + B\tilde{u}_0 = \bar{0}$, which yields (3.19). The theorem is proved.

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