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Mutual singularities of overlapping attractor and repeller

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ABSTRACT

We apply the concepts of relative dimensions and mutual singularities to characterize the fractal properties of overlapping attractor and repeller in chaotic dynamical systems. We consider one analytically solvable example (a generalized baker's map); two other examples, the Anosov–Möbius and the Chirikov–Möbius maps, which possess fractal attractor and repeller on a two-dimensional torus, are explored numerically. We demonstrate that although for these maps the stable and unstable directions are not orthogonal to each other, the relative Rényi and Kullback–Leibler dimensions as well as the mutual singularity spectra for the attractor and repeller can be well approximated under orthogonality assumption of two fractals.

Strange attractors and repellers in dissipative dynamical systems are fractals, properties of which can be characterized with generalized dimensions and singularity spectra. If these two fractals have a common support, one has a situation where two fractal distributions overlap. To characterize the properties of such an overlap, concepts of relative dimensions and mutual singularities have been introduced in the literature. We apply these concepts to characterize attractors and repellers in modifications of Anosov and Chirikov maps, where non-conservation of the phase volume, responsible for fractality, is introduced by virtue of the Möbius map (MM).

I. INTRODUCTION

In the studies of dissipative dynamical systems, notions of attractors and repellers are principal; moreover, there are many definitions relevant in different situations. In many cases, attractors are isolated fractal sets, which attract in course of time all the points from the basin.[1] Repellers are attractors in reverse time; they are nontrivial if the phase space is compact[2] so that a trajectory in backward time does not go to infinity. A compact phase space is characteristic for systems where all the variables are phases and/or angles. Another typical situation where repellers are nontrivial is that of time-reversible systems.[3–6]

Although usually attractors and repellers do not overlap, there are cases where they have a common support.[7] Recently, such a situation, called mixed dynamics, attracted much interest both from the viewpoint of mathematical theory of dynamical systems and in numerical explorations of the dynamics (see Refs. 8–11, and references therein). Typically, in such situations, the attractor and the repeller have a common support, which is non-fractal, but physically relevant invariant measures forward and backward in time are two different fractal measures. Below, we will call these two measures attractor and repeller. Both of them are multifractals, and one faces a problem of characterizing them with mutual dimensions and a mutual singularity spectrum. This is the goal of this paper. We use examples similar to those explored in our recent publication,[12] where we studied the Kantorovich–Rubinstein–Wasserstein distance (KRWD) between overlapped attractor and repeller. This distance can be calculated for any two measures, not necessarily fractal ones, and also for those not having a common support. The calculations of mutual dimensions and singularities, however, heavily rely on the latter property.

The paper is organized as follows. In Sec. II, we introduce several existing concepts of relative dimensions and mutual singularities of two fractal measures. Next, in Sec. III, we discuss a solvable example—a two-dimensional baker's map. After that, in Sec. IV, we introduce two nontrivial examples of overlapping attractor and repeller. In both cases, we start with an area-preserving map on a two-dimensional torus and incorporate dissipation via superposition with a Möbius map.[13,14] In one example, we start with a hyperbolic Anosov map and in another case with a non-hyperbolic standard Chirikov map. For these two cases, we calculate different variants of relative dimensions and mutual singularities. We discuss the results in Sec. V.

II. RELATIVE DIMENSIONS AND MUTUAL SINGULARITIES

A. Generalized dimensions and singularity spectrum of one fractal measure

This theory is well-established, and here we just present the main expressions for completeness of presentation. We consider a set U with a fractal measure. Covering the set with boxes of size ϵ, we get a finite-size approximation to the fractal measure, with measures of boxes u_i (normalization requires $\sum_i u_i = 1$). Going to finer partitions, one defines quantities $\tau(q;U)$ and generalized dimensions $D(q;U)$ according to

$$\tau(q;U) = \lim_{\epsilon\to 0}\frac{\ln\sum_i u_i^q}{\ln\epsilon},\quad D(q;U) = \frac{\tau(q;U)}{q-1} = \lim_{\epsilon\to 0}\frac{1}{q-1}\frac{\ln\sum_i u_i^q}{\ln\epsilon}. \tag{1}$$

Note that the sum in (1) can be represented as an average over finite-size boxes $\sum_i u_i^q = \langle u^{q-1}\rangle_U$. Most important are: the box-counting dimension $D(0;U)$ (gives the number of voids); the information dimension $D(1;U)$ [gives the averaged crowding index; see Eq. (3) below]; and the correlation dimension $D(2;U)$ (easily calculated with the Grassberger–Procaccia method[15]).

The singularity spectrum $f(\alpha;U)$ is defined via the Legendre transformation of $\tau(q;U)$,

$$f(\alpha;U) = \alpha q - \tau(q;U),\quad \alpha = \frac{d\tau}{dq}. \tag{2}$$

The crowding index α has the meaning of a local (in a vicinity of some point x of the fractal) scaling relation for the measure $u(\epsilon,x) \sim \epsilon^{\alpha(x)}$. The spectrum $f(\alpha)$ has the meaning of a box-counting dimension of the set of boxes having index α, and the number of these boxes scales as $N(\alpha) = \epsilon^{-f(\alpha)}$. Rewriting the sum in (1) as $\sum_i u_i^q = N(\alpha)\epsilon^{q\alpha} = \epsilon^{q\alpha - f(\alpha)}$ and calculating it at a maximum (which is justified by the limit $\epsilon \to 0$) yields Legendre transform (2). Notice that the averaged crowding index is exactly the information dimension

$$\langle\alpha\rangle_U = \sum_i u_i\alpha_i = \lim_{\epsilon\to 0}\sum_i u_i\frac{\log u_i}{\log\epsilon} = D(1;U). \tag{3}$$

B. Relative dimensions based on Rényi and Kullback–Leibler divergences

Rényi divergence[16] characterizes the distance between two fractal measures U and V, *having a common support*. We introduce it following Ref. 17. Given two measures U and V, with corresponding values in ϵ-boxes u_i and v_i, the Rényi divergence is defined as

$$R(\epsilon,q;U||V) = \frac{1}{q-1}\ln\sum_i u_i^q v_i^{1-q} = \frac{1}{q-1}\ln\left\langle\left(\frac{u_i}{v_i}\right)^{q-1}\right\rangle_U. \tag{4}$$

Here, the index at the averaging sign indicates that the averaging is over measure U.

Then, one can define the relative Rényi dimension as

$$D^R(q;U||V) = \lim_{\epsilon\to 0}\frac{R(\epsilon,q;U||V)}{\ln\epsilon} = \lim_{\epsilon\to 0}\frac{1}{q-1}\frac{\ln\sum_i u_i^q v_i^{1-q}}{\ln\epsilon} = \lim_{\epsilon\to 0}\frac{1}{q-1}\frac{\ln\left\langle\left(\frac{u_i}{v_i}\right)^{q-1}\right\rangle_U}{\ln\epsilon}. \tag{5}$$

It has following properties:

1. The Rényi dimension is in general asymmetric but one has a relation
$$qD^R(1-q;U||V) = (1-q)D^R(q;V||U).$$
2. If one of the measures is uniform A (we assume a d-dimensional object here, where d is an integer), then the relative Rényi dimension to the uniform distribution is
$$D^R(q;U||A) = D(q;U) - d,$$
where $D(q;U)$ is the introduced above generalized dimension of measure U. One can also write $D(q;U) = d + D^R(q;U||A)$. This means that dimension of U is dimension of A plus relative Rényi dimension (the latter can be negative).
3. If two measures coincide, then $D^R(q;U||V) = D^R(q;V||U) = 0$.
4. If $q \to 1$, then the Rényi divergence goes to the Kullbak-Leibler divergence, and correspondingly one obtains, dividing by $\log\epsilon$ and taking the limit, the Kullback–Leibler relative dimension,
$$D^R(1;U||V) = \lim_{\epsilon\to 0}\sum_i u_i\frac{\ln\frac{u_i}{v_i}}{\ln\epsilon} = D^{KL}(U||V). \tag{6}$$
5. The relative Kullback–Leibler dimension (6) is an analog of the information dimension for a single measure (3). Suppose that the scalings at point i are $u_i \sim \epsilon^{\alpha_i}$ and $v_i \sim \epsilon^{\beta_i}$. Then,
$$D^{KL}(U||V) = \langle\alpha_i - \beta_i\rangle_U,\quad D^{KL}(V||U) = \langle\beta_i - \alpha_i\rangle_V.$$
6. $D^R(0;U||V) = 0$.

7. Symmetric case $q = 1/2$ yields the "Hellinger mutual dimension" (based on the Hellinger distance[17])

$$D(1/2; U||V) = -2 \lim_{\epsilon \to 0} \frac{\ln(1 - \frac{1}{2} Hel^2(U,V))}{\ln \epsilon},$$
$$Hel^2(U,V) = \sum_i (u_i^{1/2} - v_i^{1/2}).$$

C. Mutual singularity spectrum

The mutual (joint) singularity spectrum has been presumably first introduced by Meneveau *et al.*[18] (see also a similar approach in Ref. 19). Similar to the Rényi dimension, one considers two measures and a generalized scaling relation like (5) but with two generally independent powers q and p,

$$\mathcal{T}(q,p;U,V) = \lim_{\epsilon \to 0} \frac{\ln \sum_i u_i^q v_i^p}{\ln \epsilon} = \lim_{\epsilon \to 0} \frac{\ln \langle u_i^{q-1} v_i^p \rangle_U}{\ln \epsilon} = \lim_{\epsilon \to 0} \frac{\ln \langle u_i^q v_i^{p-1} \rangle_V}{\ln \epsilon}. \quad (7)$$

This quantity has following properties:

1. In general, one cannot define a generalized dimension depending on two indices q, p. However, for $p = 1 - q$, one obtains the same expression as in the definition of the Rényi dimension.
2. In the case $q = p = 0$, one obtains $\sum_i u_i^0 v_i^0$ (the number of common non-empty boxes) so that $\mathcal{T}(0,0;U,V) = -D(0; U \cap V)$ is the box-counting dimension of the overlap of supports of U and V.
3. In the case $q = p = 1$, the quantity $\sum_i u_i v_i$ can be considered as a correlation of two sets and evaluated from a time series of length N (similarly to the Grassberger–Procaccia method for the correlation dimension[15]) as
$$C(U,V,\epsilon) = \frac{1}{N^2} \text{ (number of pairs with distance } < \epsilon).$$
This yields the cross-correlation dimension
$$D^K(U,V) = \mathcal{T}(1,1;U,V) \quad (8)$$
of two measures, discussed by Kantz.[20] We stress that the calculation of $D^K(U,V)$ can be performed directly from the trajectories, without estimating measures of the boxes u_i, v_i. This allows for an extra check of the validity of numerical evaluation of $\mathcal{T}(q,p;U,V)$ (at least at one set of indices q,p). Furthermore, this quantity could be potentially easier to calculate for continuous-time chaotic systems, where the capacity dimensions of attractors and repellers are at least three.
4. In the case where $p = 0$, one obtains
$$\mathcal{T}(q,0;U,V) = \lim_{\epsilon \to 0} \frac{\ln \sum_i u_i^q}{\ln \epsilon} = (q-1)D(q;U),$$
where $D(q;U)$ is the generalized dimension of measure U.
5. In the case where V is non-fractal (i.e., equivalent to Lebesgue measure), one obtains $\mathcal{T}(q,p;U,V) = pd + \tau(q;U)$, where d is the integer dimension of V.

The mutual singularity spectrum is introduced by using two crowding indices $u_i \sim \epsilon^{\alpha_i}$ and $v_i \sim \epsilon^{\beta_i}$ and defining the number of boxes with index pair (α, β) as $N(\alpha,\beta) \sim \epsilon^{-F(\alpha,\beta)}$. Then, the sum in (7) can be evaluated as $\sim \iint d\alpha d\beta \epsilon^{q\alpha + p\beta - F(\alpha,\beta)}$. Asymptotic evaluation of this integral at $\epsilon \to 0$ leads to a Legendre transform from $\mathcal{T}$ to F,

$$\mathcal{T}(q,p;U,V) + F(\alpha,\beta;U,V) = q\alpha + p\beta, \quad \alpha = \frac{\partial \mathcal{T}}{\partial q}, \; \beta = \frac{\partial \mathcal{T}}{\partial p}. \quad (9)$$

D. Riedi–Scheuring relative dimension

Yet another characterization of relative singularities has been suggested by Riedi and Scheuring[21] (see also Ref. 22). Here, first the relative partition function

$$S(q,t;U||V) = \sum_i u_i^q v_i^{-t} = \left\langle \frac{u^{q-1}}{v^t} \right\rangle_U \quad (10)$$

is defined. The condition that this function remains constant as $\epsilon \to 0$ defines the particular value of parameter t: $t = T(q;U||V)$. From this quantity, the relative dimension

$$D^{RS}(q;U||V) = \frac{T(q;U||V)}{q-1} \quad (11)$$

is defined.

Comparing expressions (10) and (7), one finds that $T(q;U||V)$ is the root of the equation

$$\mathcal{T}(q, -T(q;U||V)) = 0. \quad (12)$$

Because Riedi–Scheuring characteristics does not provide additional information compared to the calculation of $\mathcal{T}(q,p;U,V)$, we will not follow it below.

E. "Orthogonal" fractal sets

Below, we apply the introduced characterizations of two fractal sets to attractors and repellers in two-dimensional maps. For an attractor, the SBR (Sinai–Bowen–Ruelle) invariant measure is continuous along the unstable direction and fractal along the stable one; for the repeller, these directions exchange. It is instructive to consider first an ideal case where the fractal directions of sets U and V are strictly orthogonal. Furthermore, because the concepts above are applicable to sets with a common support, we consider two measures having box-counting dimension equal to the full dimension of

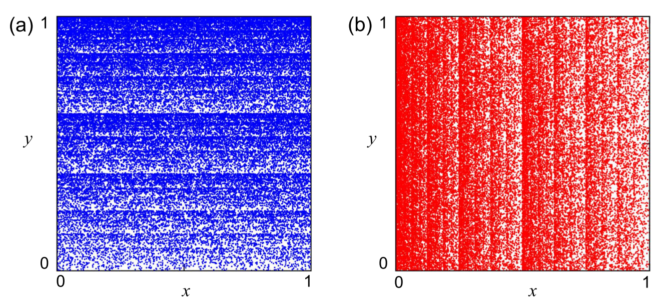

FIG. 1. Images of attractor [panel (a)] and repeller [panel (b)] of the baker's map with $\alpha = 1/2$ and $\gamma = (\sqrt{5} - 1)/2$.

the phase space (in our case 2). In other words, these sets have no voids (which are characteristic features of standard Cantor sets) but their measures are multifractals.

Therefore, we assume that on a unit square, measure U is fractal along the x axis (and we denote the projection of the measure on the x axis as μ) and uniform along the y axis. Measure V is assumed to be fractal along the y axis (and we denote the projection on the y axis as ν) and uniform along the x axis. The measures of a two-dimensional box with indices (i, j) of size ϵ are $u_{ij} = \mu_i \epsilon$ and $v_{ij} = \nu_j \epsilon$.

The fractal dimensions of the measures are obtained by inserting these expressions to (1),

$$\begin{aligned} \tau(q; U) &= \lim_{\epsilon \to 0} \frac{\ln \sum_i \mu_i^q + (q-1)\ln\epsilon}{\ln\epsilon} \\ &= \tau(q;\mu) + q - 1, \quad D(q; U) = D(q;\mu) + 1, \\ \tau(p; V) &= \lim_{\epsilon \to 0} \frac{\ln \sum_j \nu_j^p + (p-1)\ln\epsilon}{\ln\epsilon} \\ &= \tau(p;\nu) + p - 1, \quad D(p; V) = D(p;\nu) + 1. \end{aligned} \tag{13}$$

We stress here that because the supports of two measures are the full square, $D(0;\mu) = D(0;\nu) = 1$.

A similar calculation of the Rényi relative dimension yields

$$\begin{aligned} D^R(q; U||V) &= D(q;\mu) + \frac{qD(1-q;\nu) - 1}{1-q} \\ &= D(q; U) + \frac{qD(1-q; V) - 2}{1-q}. \end{aligned} \tag{14}$$

To obtain the Kullback–Leibler relative dimension one has to take the limit $q \to 1$,

$$D^{KL}(U||V) = D(1; U) - \tau'(0; V). \tag{15}$$

Evaluation of expression (7) is also straightforward,

$$\mathcal{T}(q, p; U, V) = q + p + \tau(q;\mu) + \tau(p;\nu) = \tau(q; U) + \tau(p; V) + 2. \tag{16}$$

From this formula, it follows that the cross-correlation dimension (8) independently of the fractal properties of the measures is equal

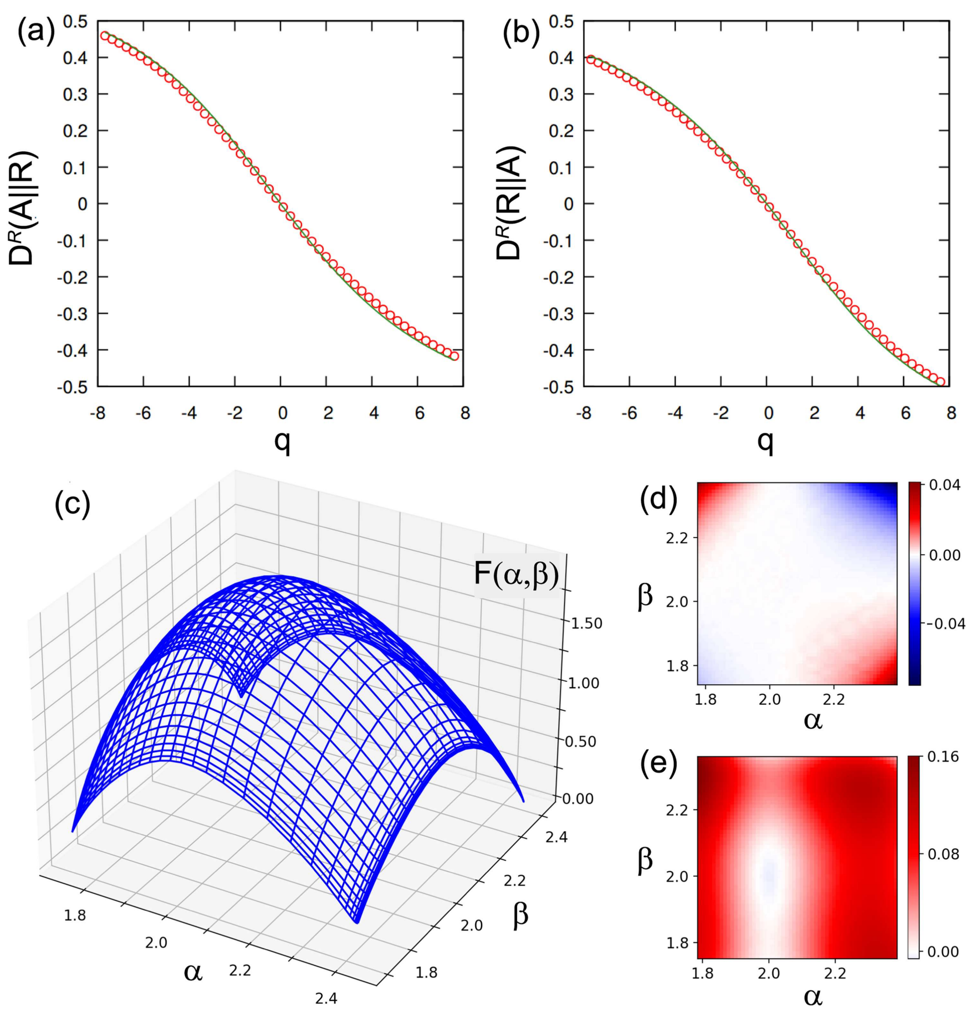

FIG. 2. Panels (a) and (b) show relative Renyi dimensions for the baker's map (markers: numerical values; lines: analytic expressions): (a) $D^R(q; A||R)$, (b) $D^R(q; R||A)$. Panel (c) shows singularity spectrum $F(\alpha, \beta)$ for this map. Color panels (d) and (e) show deviations of the numerically obtained values from the combination of separately numerically obtained partial singularity spectra $f(\alpha; A)$ and $f(\beta; R)$ [panel (d)] and from the analytic expression [panel (e)].

to 2. In fact, Kantz[20] argued that this is valid not only for "orthogonal" fractals but for any non-zero angle between the continuous directions. Application of the Legendre transform to (16) yields an expression for the mutual singularity spectrum in terms of partial spectra,

$$F(\alpha,\beta) = f(\alpha; U) + f(\beta; V) - 2. \tag{17}$$

Finally, the Riedi–Scheuring characteristics appears in a nontrivial way: $T(q)$ is a solution of equation

$$0 = \mathcal{T}(q, -T) = q - T + \tau(q;\mu) + \tau(-T;\nu). \tag{18}$$

III. SOLVABLE MODEL: BAKER'S MAP

In this section, we introduce a solvable example of an invertible two-dimensional map with a simple orthogonal structure of the attractor and the repeller. This map belongs to the class defined in Ref. 23. Below, we follow the variant of Kuznetsov.[24] The map is defined through the following expressions:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{x_n}{\alpha} \\ \gamma y_n \end{pmatrix}, & x_n \le \alpha, \\ \begin{pmatrix} \frac{x_n-\alpha}{1-\alpha} \\ 1 + (1-\gamma)(y_n - 1) \end{pmatrix}, & x_n > \alpha. \end{cases} \tag{19}$$

The dimensions of the attractor of this map can be obtained by virtue of consideration of the partition function (see details in Ref. 25), and lead to the following equation for $\tau(q; A)$:

$$\frac{\alpha^q}{\gamma^{\tau-q+1}} + \frac{(1-\alpha)^q}{(1-\gamma)^{\tau+q-1}} = 1. \tag{20}$$

For the dimensions of the repeller, one should consider the inverse map, and the corresponding expression is the same as (20) with replacement $\alpha \leftrightarrow \gamma$.

For some values of α and γ, it is possible to obtain an explicit formula for the dimensions. We set $\alpha = 1/2$ and $\gamma = (\sqrt{5}-1)/2$ so that $1-\gamma = \gamma^2$. The attractor and the repeller for this case are illustrated in Fig. 1. Then, Eq. (20) for the attractor reads

$$\left(\frac{1}{2}\right)^q \left(\frac{1}{\gamma}\right)^{\tau-q+1} + \left(\frac{1}{2}\right)^q \left(\frac{1}{\gamma}\right)^{2(\tau-q+1)} = 1,$$

which is an easily solvable quadratic equation. The expressions for the quantities $\tau(q; A)$ and $D(q; A)$ are

$$\tau(q; A) = q - 1 - \frac{\log\left(\sqrt{1/4 + 2^q} - 1/2\right)}{\log\gamma},$$

$$D(q; A) = 1 - \frac{\log\left(\sqrt{1/4 + 2^q} - 1/2\right)}{(q-1)\log\gamma}. \tag{21}$$

An equation for $\tau(p; R)$ for the repeller, following from (20), is

$$\gamma^p 2^{\tau-p+1} + \gamma^{2p} 2^{\tau-p+1} = 1,$$

which yields

$$\tau(p; R) = p - 1 - \frac{\log(\gamma^p + \gamma^{2p})}{\log 2}, \quad D(p; R) = 1 - \frac{\log(\gamma^p + \gamma^{2p})}{(p-1)\log 2}. \tag{22}$$

With these expressions, one can find mutual singularities by applying expressions from Sec. II E.

We compare the analytic solutions for the Rényi relative dimensions and for the mutual singularities in Fig. 2. Panels (a) and (b) show relative Renyi dimensions $D_q(A||R)$ and $D_q(R||A)$. Here, solid lines correspond to the analytical results, while markers are used for the numerically obtained values. One can see that the numerical values slightly deviate from the theory for large in absolute value indices q. This is a well-known pitfall in the direct computations of the generalized dimensions because the cells possessing small measure lead to large errors if a large negative power of this measure is calculated. These numerical errors also explain why the numerically obtained mutual singularity spectrum $F(\alpha,\beta)$ presented in Fig. 2(c) is closer to the combination of numerically obtained partial spectra $f(\alpha; A)$ and $f(\beta; R)$ [see the comparison chart in Fig. 2(d)], rather than to the analytic expression [this comparison chart is shown in Fig. 2(e)]. The reason is that in both calculations the numerical errors are at the same places (cells with small measure) and, therefore, yield deviations in the same direction.

IV. ATTRACTOR AND REPELLER IN ANOSOV–MÖBIUS AND CHIRIKOV–MÖBIUS MAPS

A. Anosov–Möbius map

Our first model for nontrivial overlapping attractor and repeller is the Anosov–Möbius (AM) map introduced in Ref. 12. The starting point is the Anosov cat map A,

$$\begin{aligned} x_{n+1} &= 2x_n + y_n \pmod 1, \\ y_{n+1} &= x_n + y_n \pmod 1 \end{aligned} \tag{23}$$

of a unit torus. This is a seminal example of conservative hyperbolic chaos,[26] the dynamics is fully invertible, and the attractor and repeller coincide. To "split" the attractor and the repeller, one has to introduce dissipation. To be more precise, one has to introduce non-conservation of the phase volume, i.e., at some places of the torus, the Jacobian of the transformation should be less than one, while at other domains, it should be larger than one. Sinai suggested to add a term $\sim \varepsilon \sin 2\pi x_n$ to the right hand side of equation for x_{n+1} in (23). This map was explored in Refs. 23 and 25 and in a prominent study by Anishchenko and collaborators.[27] Such a map is, however,

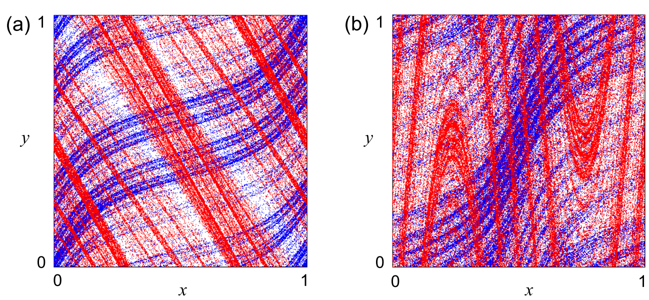

FIG. 3. Images of the attractor (blue points) and the repeller (red points) in (a) Anosov–Möbius map (28) and (b) Chirikov–Möbius map (30). In both cases, $\varepsilon = 0.4$.

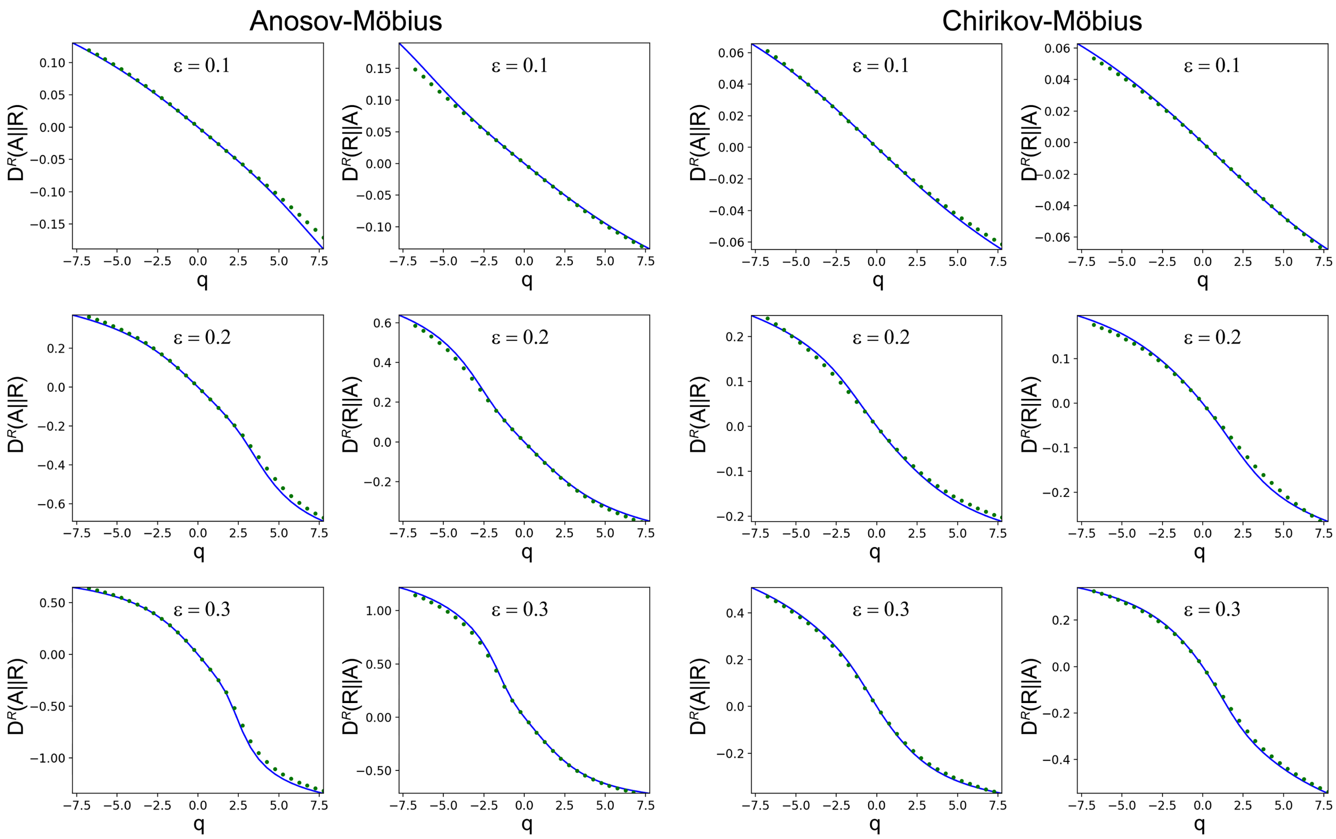

FIG. 4. Relative Rényi dimensions for the Anosov–Möbius map (left two columns) and the Chirikov–Möbius map (right two columns); odd columns: $D^R(q; A||R)$, even columns: $D^R(q; R||A)$. The rows from top to bottom: dissipation constants in the Möbius map $\varepsilon = 0.1,\ 0.2,\ 0.3$. Blue curves: direct calculations of the dimensions; green circles: the values obtained from partial dimensions using the orthogonality relation (14). One can see that this relation works well in all cases. Notice that in all cases, $D^R(q = 0) = 0$, as it should be for two measures having the same support (see property 6 in the discussion of the Rényi dimensions).

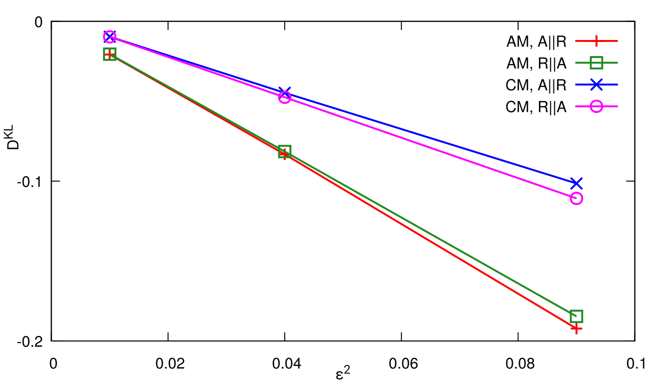

FIG. 5. Relative Kullback–Leibler dimensions for the Anosov–Möbius (AM) and Chirikov–Möbius (CM) maps. The horizontal axis is ε^2, to make evidence that $D^{KL} \sim \varepsilon^2$.

not easily numerically invertible. Therefore, we combine it with the Möbius map.[13,14]

The Möbius map (MM) is a circle map $x_n \to x_{n+1} \pmod 1$ depending on three parameters $0 \le u, v < 1$, and $-1 < \varepsilon < 1$,

$$e^{i2\pi(x_{n+1}-v)} = \frac{\varepsilon + e^{i2\pi(x_n-u)}}{\varepsilon e^{i2\pi(x_n-u)} + 1}. \tag{24}$$

For a numerical implementation, it is convenient to rewrite this formula in the real form,

$$\cos(2\pi(x_{n+1}-v)) = \frac{(1+\varepsilon^2)\cos(2\pi(x_n-u)) + 2\varepsilon}{1 + 2\varepsilon\cos(2\pi(x_n-u)) + \varepsilon^2},$$

$$\sin(2\pi(x_{n+1}-v)) = \frac{(1-\varepsilon^2)\sin(2\pi(x_n-u))}{1 + 2\varepsilon\cos(2\pi(x_n-u)) + \varepsilon^2},$$

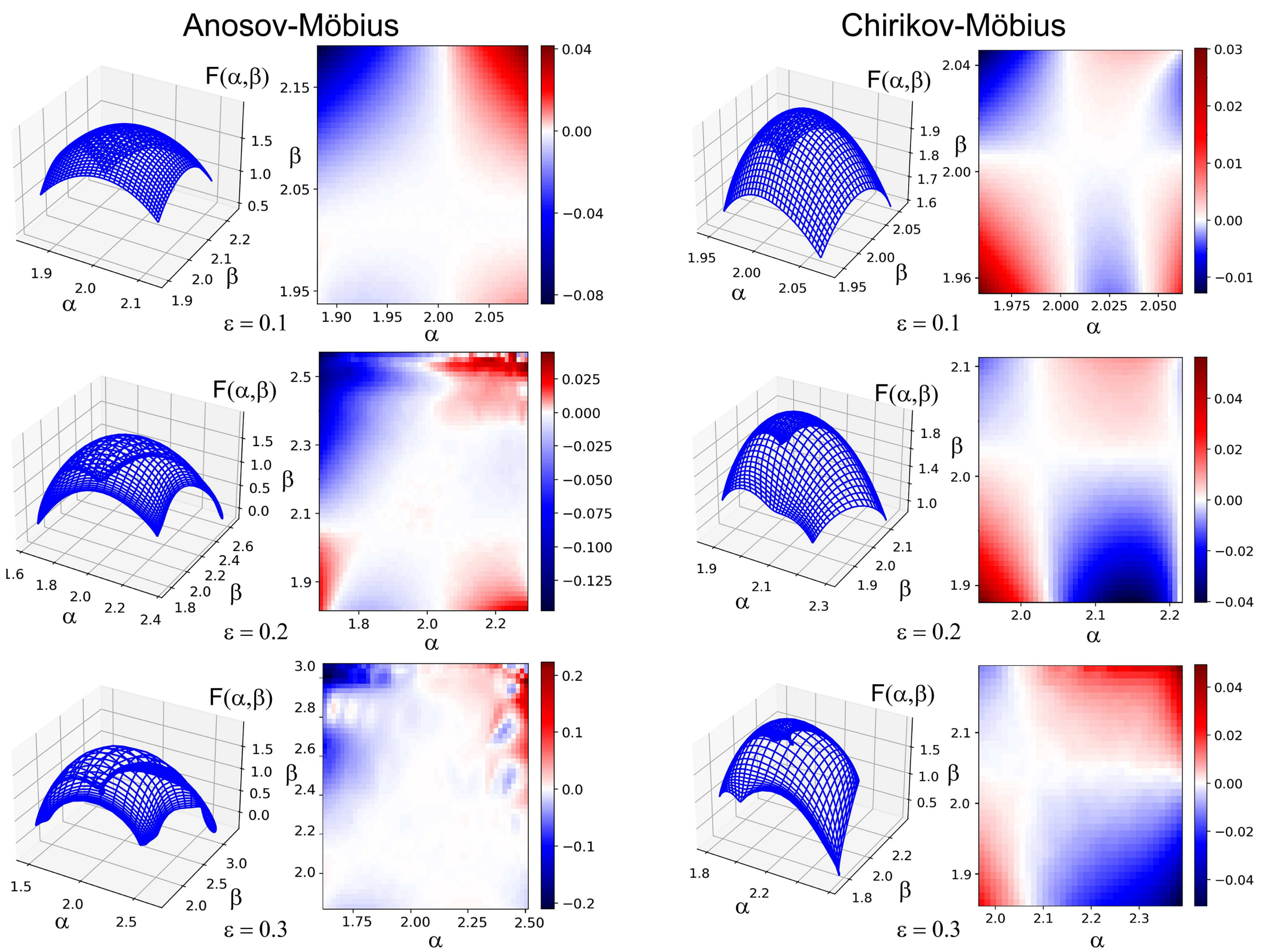

FIG. 6. Singularity spectrum $F(\alpha, \beta)$ for the Anosov–Möbius map (left two columns) and the Chirikov–Möbius map (right two columns) for the same values of parameter ε as in Fig. 4. Charts in even columns show errors of the representation of the mutual singularity spectrum through partial spectra according to expression (17).

and to express x_{n+1} from these equations using the argument of the complex number,

$$x_{n+1} = v + \frac{1}{2\pi}\arg((1+\varepsilon^2)\cos(2\pi(x_n - u)) + 2\varepsilon + \mathrm{i}(1-\varepsilon^2)\sin(2\pi(x_n - u))). \tag{25}$$

Parameter ε determines level of contraction on the circle: for $\varepsilon = 0$, the MM is a circle shift; for $\varepsilon \to 1$, it maps almost all circle to a small neighborhood of one point on it.

The MM is invertible, its inverse map, as one can easily see from the following representation

$$M(\varepsilon, u, v) : \tan(\pi(x_{n+1} - v)) = \frac{1-\varepsilon}{1+\varepsilon}\tan(\pi(x_n - u)) \tag{26}$$

is also a Möbius map

$$M^{-1}(\varepsilon, u, v) = M(-\varepsilon, v, u). \tag{27}$$

Finally, we apply the MM with $u = v = 0$ in composition with map A,

$$\mathcal{A} : \begin{pmatrix} M_\varepsilon & 0 \\ 0 & 1 \end{pmatrix} A. \tag{28}$$

Dissipation introduced by the Möbius map splits attractor and repeller, as is illustrated in Fig. 3(a).

To apply the multifractal characterization as described above, we need to be sure that the attractor and the repeller have a common support. There is an argument, attributed in Refs. 23, 25, and 27 to Sinai, that due to hyperbolicity of the Anosov map, for small perturbations, the support of the measure is not changed; thus, both attractor and repeller are dense on the torus (and have box-counting dimensions 2). We confirmed this in the numerics below by checking that no one of the small boxes in the finest partition used (in our case 2048 × 2048) is empty. However, for larger values of ε, such a procedure might be problematic because some boxes with exponentially small probability may appear as empty for a finite length of a trajectory.

B. Chirikov–Möbius map

Another map we explore below is the Chirikov standard map C,

$$\begin{aligned} x_{n+1} &= x_n + K\sin(2\pi y_n) \quad (\text{mod } 1), \\ y_{n+1} &= y_n + x_{n+1} \quad (\text{mod } 1). \end{aligned} \tag{29}$$

It is a basic example of nonhyperbolic Hamiltonian dynamics with a divided phase space.[28] For $K \gg 1$, the dynamics is predominantly chaotic. Below, we adopt $K = 14/(2\pi)$. Finally, we combine the Chirikov map with the Möbius map

$$\mathcal{C} : \begin{pmatrix} M_\varepsilon & 0 \\ 0 & 1 \end{pmatrix} C, \tag{30}$$

what begets fractal attractor and repeller as presented in Fig. 3(b).

In contradistinction to the Anosov map, the Chirikov map is generally not ergodic, as small islands filled with elliptic orbits cannot be excluded. Thus, one can hardly make general statements about the support of attractors and repellers. For the parameters under consideration, we checked that the support is full torus numerically by inspecting non-emptyness of all the fine-grid boxes of the partition used.

C. Rényi and Kullback–Leibler dimensions

We show relative Rényi dimensions of attractors and repellers of the Anosov–Möbius map and the Chirikov–Möbius map in Fig. 4, for three values of parameter ε: $\varepsilon = 0.1, 0.2, 0.3$. We remind that for $\varepsilon = 0$, the attractor and the repeller coincide so that the relative dimension vanishes. The first observation is that dimensions are indeed asymmetric: $D^R(q; A||R) \neq D^R(q; R||A)$ (cf. the first and second columns for the Anosov–Möbius map and the third and fourth columns for the Chirikov–Möbius map). The range of relative dimensions grows with the dissipation parameter ε. At the same panels, we show the relative dimensions obtained from direct calculations (blue-colored curves) and calculated from partial dimensions of the attractor and the repeller by virtue of expression (14) (green circles). One can see that in all situations, these values are very close.

The relative Kullback–Leibler dimensions for both the Anosov–Möbius and Chirikov–Möbius maps are presented in Fig. 5. Our calculations show that in the explored range of dissipative constants ε a relation $D^{KL} \sim \varepsilon^2$ holds.

D. Spectrum of singularities

We show the spectra of the mutual singularities $F(\alpha, \beta)$ for the Anosov–Möbius and Chirikov–Möbius maps, for the same parameters as in Fig. 4, in Fig. 6. The spectrum is wider for larger values of the dissipation constant ε; otherwise, no qualitative changes are observed. Panels in even columns in Fig. 6 show errors of the representation of the mutual singularity spectrum through partial spectra according to expression (17). The errors are rather small, except for the indices at the borders of the support of the spectrum, indicating that approximation of "orthogonality" is well justified for the cases considered.

V. CONCLUSION

In this paper, we studied relative dimensions and mutual singularities of overlapping attractors and repellers. Such attractors and repellers are typical for systems given on compact manifolds close to conservative ones. In the conservative case, the invariant measures forward and backward in time coincide but with the addition of a small non-conservative perturbation (which leads to shrinking of the phase volume in some parts of the phase space and to the expansion of this volume in other parts) two natural invariant measures for forward and backward evolution become different and fractal. Nevertheless, these measures have a common support (at least for a weak non-conservation of the phase volume, one can expect that both attractor and repeller have capacity dimension equal to the full dimension of the phase space), and this property allows us to apply the concepts of relative dimensions and mutual singularity spectra. Formally, these concepts are also applicable to two measures with non-integer capacity dimensions, i.e., to fractals that have voids. However, we do not know any example of attractor and repeller having such a property.

In the literature, there are different approaches to relative dimensions. Above, we concentrated on the Rényi dimension (which include also Kullback–Leibler dimension as a particular case) and on the mutual singularity spectrum according to Meneveau *et al.*[18] (which also includes the cross-correlation dimension discussed by Kantz[20] as a particular case). Our results show that the range of relative dimensions and mutual singularities grows with the parameter of non-conservation of the phase volume, as one could expect. It appears that the most convenient characteristic of discrepancy between attractor and repeller (in fact, two characteristics) is the Kullback–Leibler dimension. As has been shown in this paper, it vanishes if the attractor and the repeller coincide and grows $\sim \varepsilon^2$ if a dissipation $\sim \varepsilon$ is added to the conservative dynamics. Another possible candidate for characterization of the divergence, the cross-correlation dimension (which is much simpler to calculate compared to the Kullback–Leibler dimension), is in fact not sensitive to the difference of the sets and thus not useful (see also discussion below).

Another finding is that for all considered cases one can, with good accuracy, represent relative dimensions and mutual singularities from the separate characteristics of the attractor and the repeller. While such a representation is theoretically justified, strictly speaking, for "orthogonal" fractals only, calculations show that deviations due to non-orthogonality are small—both in the hyperbolic case (Anosov–Möbius map), where stable and unstable directions (along which the measures are continuous) intersect transversally, and in the non-hyperbolic case (Chirikov–Möbius map), where there are tangencies of these directions. This finding is in line with the observation[20] that the cross-correlation dimension of two two-dimensional fractal sets is two if the angle between the corresponding directions of continuity of measures is non-zero (in fact, this relation was confirmed in all cases we considered, with very high

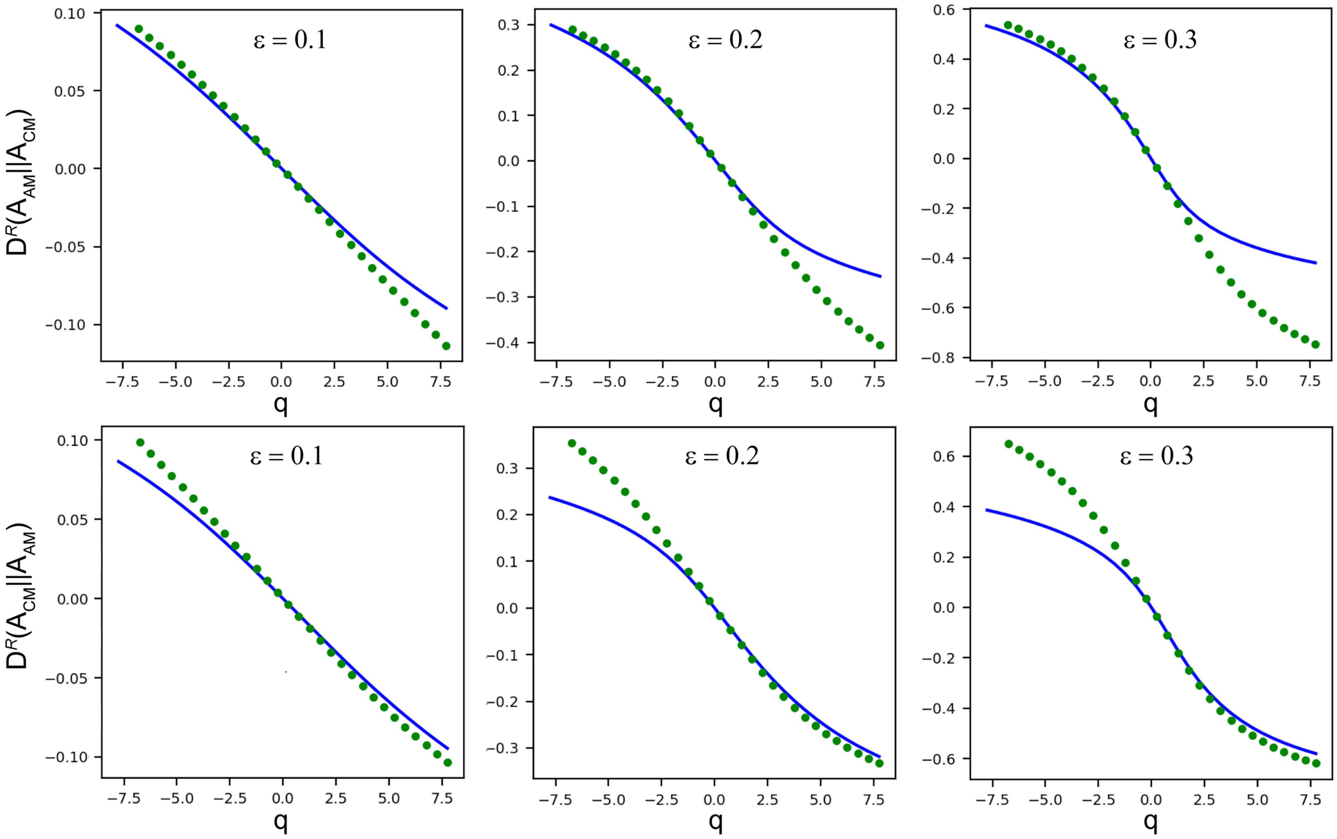

FIG. 7. Relative Rényi dimensions for the attractors of the Anosov–Möbius map and the Chirikov–Möbius map for different values of parameter ε. Blue curves: direct calculations of the dimensions; green circles: the values obtained from partial dimensions using the orthogonality relation (14). One can see that the discrepancy is much larger than in Fig. 4.

accuracy). One can conclude that in the systems considered, properties of mutual dimensions and singularities of the attractor and the repeller can be well approximated via the separate dimensions and singularities spectra of two fractals. We stress here that this is not a generic hallmark of two overlapping fractal measures. To demonstrate this, we present in Fig. 7 the relative Rényi dimensions of the attractors in the Anosov–Möbius and the Chirikov–Möbius maps. Here, the deviation of the calculated dimensions from the predictions from "orthogonal" expressions is large. This issue definitely needs further exploration.

Finally, we stress that here we focused on chaotic attractors and repellers, which in two dimensions are fractals with one continuous direction along unstable (stable for repellers) manifolds. There are non-chaotic dynamical fractals, namely, strange non-chaotic attractors (and repellers), study of their dimensions is an ongoing project.

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DATA AVAILABILITY

All numerical experiments with two-dimensional maps are described in the paper and can be reproduced without additional information.

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