On β-function of $\mathcal{N} = 2$ supersymmetric integrable $2D$ sigma models

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Motivation

- The β-functions in QFT are known to depend on the renormalization scheme.
- In QFT's with one coupling constant we can make the β-function 2-loop exact (for example, in φ^4 theory).
- In QFT's with two or more couplings it is not known in general, whether and how it is possible to achieve such a simple form.
- It is particularly interesting to study integrable deformations of 2-dimensional sigma models, for example, η-deformed $O(N)$ ones with two couplings as they admit so-called dual description in terms of Toda-like theories.
- We know the β-function for 2-dimensional sigma models up to 4-loop order in supersymmetric and non-supersymmetric case and how it varies under scheme changes.
- For $D = 2$ target space there are much less different tensor structures and we have a hope to obtain a particularly simple expression for the β-function in some scheme.
- We know some conjectured all-loop metrics in a certain scheme for η- and 2-loop ones for λ-deformed models (Hoare et al.'19), so it could be possible to find a simple expression for higher-loop β-functions.

β-function in the φ^4 theory

- Consider the ϕ^4 scalar QFT

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{4!}\varphi^4 \ .$$

- β-function in $4-\epsilon$ dimensions ('t Hooft'72) is known up to 6-loop order (Kompaniets et al.'16) in the minimal subtraction scheme ('t Hooft'73). The expression at the 2-loop order

$$\dot{g} = -\beta(g), \quad \beta(g) = -\epsilon g + 3g^2 - \frac{17g^3}{3} + \mathcal{O}(g^4) \ .$$

- Change of regularization scheme can be effectively understood as a change of the coupling

$$g \to \tilde{g}(g) = g + \xi_1 g^2 + \xi_2 g^3 + \mathcal{O}(g^4) \ .$$

- The β-function transforms as the vector field

$$\dot{\tilde{g}} = \frac{\partial \tilde{g}}{\partial g}\dot{g} \to \tilde{\beta}(\tilde{g}) = \left(\frac{\partial \tilde{g}(g)}{\partial g}\right)^{-1} \beta(\tilde{g}(g)) \ ,$$

  where

$$\dot{g} = \frac{dg}{dt}$$

  and t is the logarithm of the renormalization scale.

Normal form of the β-function in the φ^4 theory

- For the β-function, which starts from g^2, corresponding to marginal perturbation, one has

$$\beta(g) = A_1 g^2 + A_2 g^3 + A_3 g^4 + \mathcal{O}(g^5) \to$$
$$\to \tilde{\beta}(\tilde{g}) = A_1 \tilde{g}^2 + A_2 \tilde{g}^3 + (A_3 + A_2 \xi_1 + A_1(\xi_1^2 - \xi_2))\tilde{g}^4 + \mathcal{O}(\tilde{g}^5)\ .$$

- The first two coefficients are scheme independent.

- By choosing

$$\xi_1 = \frac{A_3}{A_2}\ , \quad \xi_2 = \xi_1^2$$

  we can make the β-function at the 3rd order to be 0.

- Tuning the renormalization parameters ξ_k, one can always find the *normal form* of the β-function, i.e. the scheme, in which the β-function is 2-loop exact

$$\tilde{\beta}(\tilde{g}) = A_1 \tilde{g}^2 + A_2 \tilde{g}^3\ .$$

Sigma models in 2 dimensions

- We study 2-dimensional sigma models

$$S[\boldsymbol{X}] = \frac{1}{4\pi} \int G_{ij}(\boldsymbol{X}) \partial_a X^i \partial_a X^j \, d^2\sigma \ .$$

- The metric $G_{ij}(\boldsymbol{X})$ also depends on some parameters treated as coupling constants, which vary with the scale according to RG flow equation

$$\dot{G}_{ij} + \nabla_i V_j + \nabla_j V_i = -\beta_{ij}(G) \ .$$

- The metric β-function $\beta_{ij}(G)$ admits the covariant loop expansion

$$\beta_{ij}(G) = \beta_{ij}^{(1)}(G) + \beta_{ij}^{(2)}(G) + \beta_{ij}^{(3)}(G) + \dots \ ,$$

  where L-th loop order β-function coefficient β_{ij}^{L} belongs to the finite dimensional space of tensors with given scaling properties.

- It is convenient to have in mind that the metric is proportional to the inverse of the Planck constant, which implies the following scaling for basic tensors

$$G_{ij} \sim \hbar^{-1} \to G^{ij} \sim \hbar \ , \ \Gamma_{ij}^{k} \sim \hbar^0 \ , \ \nabla_i \sim \hbar^0 \ , \ R_{ijk}{}^{l} \sim \hbar^0 \ , \ R_{ij} \sim \hbar^0 \ , \ R \sim \hbar \ .$$

β-function of 2-dimensional sigma model

- The 1-loop β-function is proportional to the Ricci curvature

$$\beta_{ij}^{(1)} = R_{ij} \ ,$$

and the corresponding RG equation is the celebrated Ricci flow equation.

- Higher loop coefficients $\beta_{ij}^{(L)}$ have been calculated in the minimal subtraction scheme: in 2 loops in (Friedan'80)

$$\beta_{ij}^{(2)} = \frac{1}{2} R_{iklm} R_j{}^{klm} \ .$$

- in 3 loops in (Graham'87, Foakes, Mohammedi'87)

$$\beta_{ij}^{(3)} = \frac{1}{8} \nabla_k R_{ilmn} \nabla^k R_j{}^{lmn} - \frac{1}{16} \nabla_i R_{klmn} \nabla_j R^{klmn} - \\ - \frac{1}{2} R_{imnk} R_{jpq}{}^k R^{mqnp} - \frac{3}{8} R_{iklj} R^{kmnp} R^l{}_{mnp} \ .$$

- Also the 4-loop result has been obtained in (Jack et al.'89).
- The higher loop coefficients $\beta_{ij}^{(L)}$ for $L > 1$ are scheme dependent. They are related by covariant metric redefinitions

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} + \sum_{k=0}^{\infty} G_{ij}^{(k)} \ ,$$

where $G_{ij}^{(k)}$ is of the order $\hbar^k$.

β-function for $D = 2$ sigma models

- The β-function for the SM with *two-dimensional* target space is significantly simplified

$$\beta_{ij}^{(1)} = \frac{1}{2} R G_{ij} \ ,$$
$$\beta_{ij}^{(2)} = \frac{1}{4} R^2 G_{ij} \ ,$$
$$\beta_{ij}^{(3)} = \left(\frac{5}{32} R^3 + \frac{1}{16} (\nabla R)^2 \right) G_{ij} - \frac{1}{16} \nabla_i R \nabla_j R \ ,$$
$$\beta_{ij}^{(4)} = \left(\frac{23}{192} R^4 + \frac{2+\zeta(3)}{32} R^2 \nabla^2 R + \frac{41+12\zeta(3)}{192} R (\nabla R)^2 + \frac{1}{192} (\nabla^2 R)^2 + \right.$$
$$\left. + \frac{1}{192} (\nabla_i \nabla_j R)^2 \right) G_{ij} - \frac{\zeta(3)}{48} R^2 \nabla_i \nabla_j R - \frac{25+8\zeta(3)}{192} R \nabla_i R \nabla_j R - \frac{1}{96} (\nabla^2 R) \nabla_i \nabla_j R \ .$$

- Covariant metric redefinition is determined by several tensor structures at every order of $\hbar$

$$G_{ij}^{(0)} = c_1 R G_{ij} \ ,$$
$$G_{ij}^{(1)} = \left(c_2 R^2 + c_3 \nabla^2 R \right) G_{ij} + c_4 \nabla_i \nabla_j R \ ,$$
$$G_{ij}^{(2)} = \left(c_5 R^3 + c_6 (\nabla R)^2 + c_7 R \nabla^2 R + c_8 \nabla^2 \nabla^2 R \right) G_{ij} +$$
$$+ c_9 \nabla_i R \nabla_j R + c_{10} R \nabla_i \nabla_j R + c_{11} \nabla_i \nabla_j \nabla^2 R$$

and so on.

β-function for $D = 2$ sigma models in different schemes I

- Up to a diffeomorphism transformation after covariant redefinition metric satisfies the RG flow equation

$$\dot{\tilde{G}}_{ij} + \tilde{\nabla}_i \tilde{V}_j + \tilde{\nabla}_j \tilde{V}_i = -\tilde{\beta}_{ij}(\tilde{G})$$

with some vector field $\tilde{V}_i$ and transformed $\tilde{\beta}_{ij}$.

- 1-loop β-function $\beta_{ij}^{(1)}$ is obviously scheme-independent.
- Perturbation of the left hand side

$$\begin{aligned}\dot{\tilde{G}}_{ij} &= \dot{G}_{ij} + \dot{G}_{ij}^{(0)} + \ldots = \dot{G}_{ij} + c_1(\dot{R}G_{ij} + R\dot{G}_{ij}) + \ldots = \\ &= \dot{G}_{ij} + \frac{c_1}{2}\Delta R G_{ij} + \ldots = -\frac{1}{2}RG_{ij} + \left(-\frac{1}{4}R^2 + \frac{c_1}{2}\Delta R\right) G_{ij} + \ldots\end{aligned}$$

- Right hand side takes the form

$$\begin{aligned}\tilde{\beta}_{ij}(\tilde{G}) &= \frac{1}{2}\tilde{R}\tilde{G}_{ij} + \left(\tilde{b}_1^{(2)}\tilde{R}^2 + \tilde{b}_2^{(2)}\tilde{\Delta}\tilde{R}\right)\tilde{G}_{ij} + \tilde{b}_3^{(2)}\tilde{\nabla}_i\tilde{\nabla}_j\tilde{R} + \ldots = \\ &= \frac{1}{2}RG_{ij} + \left(\tilde{b}_1^{(2)}R^2 + \left(\tilde{b}_2^{(2)} - \frac{c_1}{2}\right)\Delta R\right) G_{ij} + \tilde{b}_3^{(2)}\nabla_i\nabla_j R + \ldots\end{aligned}$$

- Comparing two previous expressions, we find that $\tilde{b}_1^{(2)} = \frac{1}{4}$, $\tilde{b}_2^{(2)} = 0$ and $\tilde{b}_3^{(2)} = 0$, therefore 2-loop β-function $\beta_{ij}^{(2)}$ is also scheme-independent.

β-function for $D = 2$ sigma models in different schemes II

- Higher order contributions to the β-function depend on the scheme, starting from the 3-loop order

$$\beta^{(3)}_{ij} = \left[\left(\frac{5}{32} + \frac{c_1 - 2c_2}{4}\right) R^3 + \left(\frac{1}{16} - \frac{c_1 - 2c_2}{2} - (c_1^2 + c_3)\right) (\nabla R)^2 - \right.$$
$$\left. -(c_1^2 + c_3) R \nabla^2 R\right] G_{ij} - \frac{1}{16} \nabla_i R \nabla_j R - \frac{c_4}{4} \nabla_i \nabla_j \left(3R^2 + 2\nabla^2 R\right) \; .$$

- Let us choose the covariant redefinition parameters to be

$$c_2 = -\frac{1}{16} + \frac{c_1}{2} \, , \quad c_3 = -\frac{c_1^2}{2} \, .$$

- We found the combination of the scheme change parameters, for which the β-function up to the 4-loop order is given by

$$\beta_{ij} = \left(\frac{R}{2} + \frac{R^2}{4} + \frac{3R^3}{16} + \frac{5R^4}{32} + \frac{2 + \zeta_3}{64} \nabla^2 \left(R^3 + 2R\nabla^2 R - \frac{1}{2}\nabla^2 R^2\right) + \right.$$
$$\left. + \ldots\right) G_{ij} - \left(\frac{1}{16} + \frac{5R}{32} + \ldots\right) \nabla_i R \nabla_j R + \ldots$$

- One can notice that parts of this expression without ζ_3 are the expansion of

$$\frac{R G_{ij}}{2(1-R)^{\frac{1}{2}}} - \frac{1}{16(1-R)^{\frac{5}{2}}} \nabla_i R \nabla_j R \, .$$

“All-loop” “sausage” metric

- In (Fateev et al.’93) there was obtained the solution of 1-loop RG flow equation, which was later identified as semiclassical η-deformed $O(3)$ metric (Hoare et al.’14) (also classically integrable (Lukyanov’12)).
- All-loop metric, however, in different scheme, was conjectured in (Hoare et al.’19).
- The metric takes the form

$$ds^2 = \frac{2\kappa}{\hbar} \frac{\left(1 - \frac{\hbar\kappa\cos^2\theta}{1-\kappa^2\sin^2\theta}\right) d\theta^2 + \cos^2\theta d\chi^2}{1-\kappa^2\sin^2\theta} ,$$

where the new couplings $\hbar$ and κ satisfy the following flow equations

$$\dot{\hbar} = 0, \quad \dot{\kappa} = \frac{\hbar(\kappa^2-1)}{2\left((1-\hbar\kappa)(1-\hbar\kappa^{-1})\right)^{\frac{1}{2}}} ,$$

and vector field has the form

$$V = \frac{\hbar}{\rho}\left\{\frac{\kappa(\kappa^2-1)\sin 2\theta}{4(1-\kappa^2\sin^2\theta)^2}, \frac{\cos^2\theta}{1-\kappa^2\sin^2\theta}\right\}, \quad \rho \overset{\text{def}}{=} \sqrt{(1-\hbar\kappa)(1-\hbar\kappa^{-1})} .$$

- We note that differential equation for κ is uniformized by ρ

$$\left(\frac{1-\rho-\hbar}{1+\rho-\hbar}\right)^{1-\hbar}\left(\frac{1+\rho+\hbar}{1-\rho+\hbar}\right)^{1+\hbar} = e^{2\hbar(t-t_0)} ,$$

which resembles the integral equation from (Fateev’19).

"All-loop" λ model metric

- There exists a solution to the 1-loop RG flow equation without any isometries

$$ds^2 = \frac{2}{\hbar}\frac{\kappa dp^2 + \kappa^{-1} dq^2}{1 - p^2 - q^2}\,, \quad \text{where} \quad \kappa = \frac{1-\lambda}{1+\lambda}\,.$$

- This metric is one-loop renormalizable with κ running according to the leading in $\hbar$ order of and the vector field given by

$$V_p = \frac{p}{1-p^2-q^2}\,, \quad V_q = \frac{q}{1-p^2-q^2}\,.$$

- We propose an $\hbar$ completion which is also two-loop exact similar to the all-loop "sausage" action

$$ds^2 = \frac{2}{\hbar}\left(\frac{\left(\kappa - \hbar\right)dp^2 + \left(\kappa^{-1} - \hbar\right)dq^2}{1-p^2-q^2} - \hbar\frac{\left(pdp + qdq\right)^2}{\left(1-p^2-q^2\right)^2}\right)\,.$$

supplemented by the following vector field

$$V_p = \frac{p\left(\frac{1-\hbar\kappa}{1-\hbar\kappa^{-1}}\right)^{\frac{1}{2}}}{1-p^2-q^2}\left(1 - \frac{\hbar}{2\kappa}\frac{1-\left(\frac{1-\kappa^2}{1-\hbar\kappa}\right)q}{1-p^2-q^2}\right)\,, \quad V_q = \{p \leftrightarrow q, \kappa \to \kappa^{-1}\}\,.$$

- Surprisingly, the parameter κ satisfies the same RG flow differential equation as for the η-deformed model.

UV limit of the "sausage" and λ model metrics

- We perform the following change of the variables and the coupling constants

$$\sin\theta = \kappa^{-1}\tanh\frac{x}{2}\,, \quad \chi = \frac{y}{2} + \frac{i}{2}\log\left(1 - \frac{1-\kappa^2}{1+\kappa^2}\cosh x\right) \quad \text{for “sausage” model}\,,$$

$$p^2 + q^2 = e^{iy}\,, \qquad \frac{p^2 - q^2}{p^2 + q^2} = \cosh x \qquad \text{for } \lambda \text{ model}\,.$$

- Then after some rescalings we obtain for the sausage sigma model

$$ds^2_{\text{sausage}} = \frac{1}{2}\left(dx^2 + dy^2\right) + \\ + \frac{B}{2A}\left(e^{\frac{x}{\sqrt{2A}}}\left(dx + i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2 + e^{-\frac{x}{\sqrt{2A}}}\left(dx - i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2\right)$$

and for the λ-deformed sigma model

$$ds^2_{\lambda} = \frac{1}{2}\left(\frac{1}{1 - e^{-i\frac{y}{\sqrt{2A+1}}}}dx^2 + \frac{1 - \frac{2A}{2A+1}e^{-i\frac{y}{\sqrt{2A+1}}}}{\left(1 - e^{-i\frac{y}{\sqrt{2A+1}}}\right)^2}dy^2\right) + \\ + \frac{\frac{B}{2A}}{1 - e^{-i\frac{y}{\sqrt{2A+1}}}}\left(e^{\frac{x}{\sqrt{2A}}}\left(dx + i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2 + e^{-\frac{x}{\sqrt{2A}}}\left(dx - i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2\right)\,.$$

UV limit of the "sausage" and λ model metrics II

- Coefficients A and B have the following $t \to -\infty$ expansion

$$A = \frac{1-\hbar}{2\hbar} + \mathcal{O}\left(e^{\frac{2\hbar(t-t_0)}{1-\hbar}}\right), \quad B = \frac{1-\hbar}{2\hbar}\hbar^{\frac{\hbar}{1-\hbar}} e^{\frac{\hbar(t-t_0)}{1-\hbar}} + \mathcal{O}\left(e^{\frac{3\hbar(t-t_0)}{1-\hbar}}\right).$$

- New parameter together with an additional imaginary shift of y coordinate

$$b \stackrel{\text{def}}{=} \sqrt{\frac{1-\hbar}{\hbar}}, \quad y \to y + \frac{i(t-t_0)}{b^2}.$$

- We derive the following UV limits of the fields

$$ds^2_{\text{sausage}} = \frac{1}{2}\left(dx^2 + dy^2\right) + $$
$$+e^{\frac{t-t_0}{b^2}}\left(e^{\frac{x}{b}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)^2 + e^{-\frac{x}{b}}\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right)^2\right) + \mathcal{O}\left(e^{\frac{3(t-t_0)}{b^2}}\right)$$

and

$$ds^2_{\lambda} = \frac{1}{2}\left(dx^2 + dy^2\right) + e^{\frac{t-t_0}{b^2}}\left(e^{\frac{x}{b}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)^2 + e^{-\frac{x}{b}}\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right)^2\right.$$
$$\left. + e^{\frac{t-t_0}{b^2}} e^{-\frac{iy}{\sqrt{1+b^2}}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right) + \mathcal{O}\left(e^{\frac{3(t-t_0)}{b^2}}\right).\right.$$

Intermediate conclusions and further motivation

Conclusions about the $D = 2$ targer space:

- We found the renormalization scheme, in which the expression for the 4-loop β-function for $D = 2$ sigma models is particularly simple.
- It was shown to be connected to the β-function in the minimal subtraction scheme in the first 4 loop orders by some covariant metric redefinition.
- We found the 4-loop solution to RG flow equation, corresponding to the η- and λ-deformed $O(3)$ sigma model, for which there exists the following invariant

$$R^3 + 2R\nabla^2 R - \frac{1}{2}\nabla^2 R^2 = \hbar^3 \left(\kappa - \kappa^{-1}\right)^2 \left(\kappa + \kappa^{-1}\right).$$

These results could be generalized in the further directions:

- We would like to generalize obtained results for higher dimensional target spaces.
- The higher loop β-functions are in general quite complicated for this case, however, in the supersymmetric space they are much simpler.
- We confine ourselves to the $\mathcal{N} = 2$ supersymmetric sigma models.
- In this case supersymmetry dictates the metric to be Kähler.

β-function of 2-dimensional supersymmetric sigma model

- The target space manifold is Käehler if there exists a tensor $f^j_{\ i}$ (the complex structure), which satisfies

$$f^i_{\ k} f^k_{\ j} = -\delta^i_{\ j} \,, \quad G_{ij} f^i_{\ k} f^j_{\ i} = G_{kl} \,, \quad \nabla_i f^j_{\ k} = 0 \,,$$

- These equations imply that the target space manifold can be covered with complex charts and in these coordinates

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K = \nabla_\mu \nabla_{\bar{\nu}} K \,.$$

- The β-function is known up to 4 loops in the $\mathcal{N} = 1$ case (Alvarez-Gaumé, Freedman, Mukhi'81, Alvarez-Gaumé'81, Grisaru, van de Ven, Zanon'86) and up to 5 loops in the $\mathcal{N} = 2$ case (Grisaru, Kazakov, Zanon'87).
- In the minimal subtraction scheme the β-function has the form

$$\beta_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} + \nabla_\mu \nabla_{\bar{\nu}} \Delta K + \mathcal{O}(\hbar^4)$$

  where

$$\Delta K = \frac{\zeta(3)}{24} R_{ijkl} R^i_{\ mn}{}^l \left(R^{jnmk} + R^{jkmn} \right) \,.$$

- In terms of Kähler potential the RG flow equation takes the form

$$\dot{K} = -\beta_K \,,$$

  where the Kähler β-function is given by

$$\beta_K = -\frac{1}{2} \log \det g + \Delta K + \mathcal{O}(\hbar^4) \,.$$

β-function of 2-dimensional supersymmetric sigma model II

- General change of scheme which preserves $\mathcal{N} = 2$ supersymmetry corresponds to the following change of the Kähler potential

$$K \to K + c_1 \log\det g + c_2 R + \underbrace{c_3 R^2 + c_4 \left(R_{ijkl}\right)^2 + c_5 \left(R_{ij}\right)^2 + c_6 \nabla^2 R}_{4 \text{ scalars of order } \hbar^2} +$$

$$+ \underbrace{c_7 R^3 + \dots + c_{23} \nabla^2 \nabla^2 R}_{17 \text{ scalars of order } \hbar^3} + \underbrace{c_8 R^4 + \dots + c_{99} \nabla^2 \nabla^2 \nabla^2 R}_{92 \text{ scalars of order } \hbar^4} + \dots$$

- To understand how the Kähler β-function changes in different schemes, we need to find the variations of the left and right hand sides of the RG flow equation.
- This requires translating the Kähler potential variation into the corresponding variation of metric, namely, deriving the formula

$$2\nabla_\mu \nabla_{\bar{\nu}} \Phi dz^\mu d\bar{z}^{\bar{\nu}} = T^{\Phi}_{ij} dx^i dx^j \,.$$

- The structures we are interested in

$$T^{\log\det g}_{ij} = -2R_{ij}, \quad T^{R}_{ij} = \nabla^2 R_{ij} + 2R_{ikjl}R^{kl} - 2R_{ik}R^k_j \,,$$

$$T^{R^2_{ij}}_{ij} = 2\nabla^k R_{il} \nabla_k R_j{}^l - 2\nabla^k R_{il} \nabla^l R_{jk} + \nabla_i R_{kl} \nabla^k R_j{}^l + \nabla_j R_{kl} \nabla^k R_i{}^l +$$

$$+ 2R^{kl} \nabla_k \nabla_l R_{ij} + 4R_{kilj} R^k{}_m R^{lm} + 2R_{klmi} R^k{}_j R^{lm} + 2R_{klmj} R^k{}_i R^{lm} \,,$$

$$T^{\nabla^2 R}_{ij} = \nabla^2 T^R_{ij} - \frac{1}{2}\left(R_i{}^k T^R_{kj} + R_j{}^k T^R_{ki}\right) + R_i{}^k{}_j{}^l T^R_{kl} \,.$$

β-function of 2-dimensional supersymmetric sigma model III

- We consider the following change of scheme

$$K \to K + c_1 \log\det g + c_2 R + c_3 \big(R_{ij}\big)^2 + c_4 \nabla^2 R\,,$$

which can be written in terms of the metric as

$$g_{ij} \to g_{ij} + c_1 T_{ij}^{\log\det g} + c_2 T_{ij}^{R} + c_3 T_{ij}^{R_{ij}^2} + c_4 T_{ij}^{\nabla^2 R}\,.$$

- Taking

$$c_2 = -c_1^2\,, \quad c_3 = -\frac{2c_1^3}{3} - \frac{5\zeta(3)}{48} \quad \text{and} \quad c_4 = -\frac{c_1^3}{3} - \frac{\zeta(3)}{48}\,,$$

one shows that in the new scheme the Kähler potential satisfies

$$\dot{K} = \frac{1}{2}\log\det g - \Delta\tilde{K} + \mathcal{O}(\hbar^4)\,,$$

- The β-function in the new scheme is represented by the formula

$$\Delta\tilde{K} = \frac{\zeta(3)}{24}\Big(5R^{ij}R^{kl}R_{ikjl} + R_{ik}{}^{mn}R^{ijkl}R_{jmln} - R_{im}{}^{kn}R^{ij}{}_{kl}R_{jn}{}^{lm} + \frac{1}{2}R_{ij}\nabla^i\nabla^j R + R^{ij}\nabla^2 R_{ij} - \frac{3}{2}\nabla_k R_{ij}\nabla^k R^{ij}\Big)\,.$$

Metrics of complete $T-$dual to $\eta-$deformed $\mathbb{CP}(n-1)$ model

- The $\eta-$deformed $SU(n)/SU(n-1)\otimes U(1)$ sigma-model has the form (Klimcik'08, Delduc, Vicedo'13)

$$\mathcal{S}=\frac{\kappa}{2\hbar}\int \mathrm{Tr}\left(\left(\mathbf{g}\partial_{+}\mathbf{g}^{-1}\right)^{(\mathrm{c})}\ \frac{1}{1-i\kappa\mathcal{R}_{\mathbf{g}}\circ\mathrm{P_c}}\ \left(\mathbf{g}\partial_{-}\mathbf{g}^{-1}\right)^{(\mathrm{c})}\right)d^2x\,,$$

where $\mathcal{R}$ is the Drinfeld-Jimbo solution to the modified YB equation $\mathbf{g}\in SU(n)$, $\mathcal{R}_{\mathbf{g}}=\mathrm{Ad}\,\mathbf{g}\circ\mathcal{R}\circ\mathrm{Ad}\,\mathbf{g}^{-1}$ and $\mathrm{P_c}$ is the projector on the coset space.
- The theory above contains both the metric and the $B-$field (except for the case of $n=2$).
- However the $T-$duality along all the isometric directions (complete $T-$dual) eliminates the $B-$field completely and, moreover, it has been shown that the corresponding geometry is Kähler (Litvinov'19, Bykov, Lust'20).
- The metric has a nice form of the flat metric perturbed by the "graviton-like" exponential terms

$$ds^2=|d\boldsymbol{z}|^2+\frac{2}{e^{nt}-1}\sum_{k=1}^{n}f_k(\boldsymbol{x})(\boldsymbol{h}_k\cdot d\boldsymbol{z})^2,\quad \boldsymbol{z}=\boldsymbol{x}+i\boldsymbol{y},$$

where $\boldsymbol{h}_k$ are the weights for $\mathfrak{sl}(n)$ and

$$f_k(\boldsymbol{x})=\sum_{l=1}^{n}e^{lt}e^{((\boldsymbol{\alpha}_k+\cdots+\boldsymbol{\alpha}_{k+l-1})\cdot\boldsymbol{x})},$$

where $\boldsymbol{\alpha}_k$ are the simple roots for $\mathfrak{sl}(n)$.

Examples of $\mathbb{CP}(1)$ and $\mathbb{CP}(2)$

- The simplest example of the solution for $n = 2$ of the RG flow equation in the new scheme is given by the well-known η-deformed "sausage" metric

$$ds^2 = \frac{2\kappa}{\hbar} \frac{d\theta^2 + \cos^2\theta d\chi^2}{1 - \kappa^2 \sin^2\theta} \ ,$$

where $\hbar$ is constant and κ satisfes the following flow equation

$$\dot{\kappa} = \frac{\hbar(\kappa^2 - 1)}{2} \ .$$

- In the case of $n = 3$ we obtain the η-deformed $\mathbb{CP}(3)$ metric. The whole expression is too cumbersome, so we write down only a part of it (the RG time is redefined)

$$ds^2 = \left(2 + \frac{2}{e^{3t} - 1}(e^t(e^{2x_1} + e^{-x_1+x_2}) + e^{2t}(e^{-2x_1} + e^{x_1+x_2})) + 2e^{3t}\right) dx_1^2 +$$
$$+ \left(2 - \frac{2}{e^{3t} - 1}(e^t(e^{2x_1} + e^{-x_1+x_2}) + e^{2t}(e^{-2x_1} + e^{x_1+x_2})) + 2e^{3t}\right) dy_1^2 + \dots$$

- In the previous formula we already substituted the solution of the equation

$$\dot{\kappa} = \frac{c_G}{4}\hbar(\kappa^2 - 1) \, ,$$

where c_G is the dual Coxeter number ($c_{SU(n)} = n$).

$SU(3)/U(2)$ λ-model example

- The action of the G/H $\lambda-$model has the form (here $g \in G$)

$$S = \int \mathrm{Tr}\Big(-\frac{1}{2}(g^{-1}\partial g)(g^{-1}\bar{\partial} g) + J\,(\mathrm{Ad}_g - 1 + \lambda\mathbb{P})^{-1}\,\bar{J}\Big) d^2\xi + S_{WZ},$$

where $\mathbb{P}$ is the projection on the coset space, in our case $SU(3)/U(2)$, and

$$J = g^{-1}\cdot\partial g\,, \quad \bar{J} = \bar{\partial} g \cdot g^{-1}\,.$$

- In the case we are interested in the metric takes the form

$$ds^2 = -\frac{e^{\beta}(-ie^{\gamma} + e^{2\alpha}\lambda)(e^{2\alpha} - ie^{\gamma}\lambda)}{2(e^{\alpha} - e^{\beta})(e^{\alpha} - e^{\beta+\gamma+\delta})(-1+\lambda^2)} d\alpha^2 + \ldots\,,$$

where λ depends on RG time t.

- It is well known that η and λ deformations of G/H sigma-model are related by Poisson-Lie duality with respect to group G and certain analytic continuation (Hoare, Seibold'17).
- There is another relation between two models noticed (Hoare, Tseytlin'15) for $SO(N+1)/SO(N)$. Namely, taking certain infinite limits of the $\lambda-$model one recovers the complete abelian $T-$dual of the $\eta-$model.

Conclusions and open problems

- We found the renormalization scheme, in which the expression for the 4-loop β-function for $\mathcal{N} = 2$ supersymmeric sigma models is particularly simple.
- It was shown to be connected to the Käehler β-function in the minimal subtraction scheme in the first 4 loop orders by some covariant metric redefinition.
- We found the 4-loop solution to RG flow equation, corresponding to the η-deformed $\mathbb{CP}(n)$ sigma model and λ-deformed $SU(3)/U(2)$ sigma model, which was also shown to contain the $\mathbb{CP}(2)$ as some special limit.
- Natural assumption would be that there exists the renormalization scheme, in which the solution of the RG flow equation is 1-loop exact at higher orders. This can be tested at the 5th loop order.
- Generalize the obtained results for higher dimensional non-supersymmetric sigma model target spaces.

Thanks for your attention!