On dual description of integrable sigma models

Based on M. Alfimov, B. Feigin, B. Hoare and A. Litvinov, JHEP**12**(2020)040
M. Alfimov and A. Litvinov, JHEP**01**(2022)043 and work in progress with B. Feigin

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Motivation

- The integrability-preserving deformations of $O(N)$ sigma models are known to admit the dual description in terms of a coupled theory of bosons and Dirac fermions with exponential interactions of the Toda type (Fateev, Onofri, Zamolodchikov'93, Fateev'04, Litvinov, Spodyneiko'18).
- On the other hand, there are known examples of the integrable superstring theories, such as type IIB $\mathrm{AdS}_5 \times \mathrm{S}^5$ (dual to $\mathcal{N} = 4$ SYM) and others, which also have integrable deformations.
- Our strategic goal is to build a similar dual description for the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ type IIB superstring (Arutyunov, Frolov et al.) and, possibly, other theories of this type.
- There are three major problems on this way:
  1. Incorporate the fermionic degrees of freedom into the construction of dual theory.
  2. Adapt the whole construction to describe the sigma models with non-compact target space.
  3. The superstring theory possesses the reparametrization symmetry and requires gauge fixing, which implies inclusion of this symmetry into the dual description.
- In this talk we are going to address the general scheme to build the dual description of the deformed $O(N)$ and $OSp(N|2m)$ sigma models.

Building of the dual model

Guiding principles to look for the dual description (Litvinov, Spodyneiko'18)

1. The theory has to be renormalizable (at least 1-loop). In the case of the deformed $O(N)$ and $OSp(N|2m)$ it can be checked by solving the RG flow equation.

2. The dual theory is found as an integrable perturbation from the special "free" point of the S-matrix and is determined by the set of screening charges, which commute with the integrals of motion in the leading order in the mass parameter

$$\left[I_k^{\text{free}}, \int e^{(\boldsymbol{\alpha}_r, \phi)} dz \right] = 0 \, .$$

3. In the case of the deformed $O(3)$ they are $e^{b\Phi + i\beta\varphi}$, $e^{b\Phi - i\beta\varphi}$, $e^{-b\Phi + i\beta\varphi}$ and $e^{-b\Phi - i\beta\varphi}$, where b is some continuous parameter and $\beta = \sqrt{1+b^2}$. Also, for instance, the two operators $e^{b\Phi + i\beta\varphi}$ and $e^{b\Phi - i\beta\varphi}$ define sine-Liouville CFT, therefore the dual description can be understood as an integrable perturbation of this CFT.

4. Our $O(N)$ and $OSP(N|2m)$ models are integrable deformations of some CFT, based on the cosets

$$\frac{\widehat{\mathfrak{so}}(N)_w}{\widehat{\mathfrak{so}}(N-1)_w} \quad \text{and} \quad \frac{\widehat{\mathfrak{osp}}(N|2m)_w}{\widehat{\mathfrak{osp}}(N-1|2m)_w} \, .$$

respectively.

CFT's defined by screening charges

- Let $\boldsymbol{\varphi}(z) = (\varphi_1(z), \dots, \varphi_N(z))$ be the $N-$component holomorphic bosonic field normalized as

$$\varphi_i(z)\varphi_j(z') = -\delta_{ij}\log(z-z') + \dots \quad \text{at} \quad z \to z',$$

  and $\vec{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_N)$ be the set of linear independent vectors.
- We define $W_{\vec{\boldsymbol{\alpha}}}$-algebra as a set of currents $W_s(z)$ of integer spins s such that

$$\oint_{\mathcal{C}_z} e^{(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi}(\xi))} W_s(z) d\xi = 0 \, , \quad r = 1, \dots, N \, .$$

- For generic $\vec{\alpha}$ there is a spin 2 current

$$W_2(z) = -\frac{1}{2}(\partial\boldsymbol{\varphi}(z)\cdot\partial\boldsymbol{\varphi}(z)) + (\boldsymbol{\rho}\cdot\partial^2\boldsymbol{\varphi}(z)) \, , \quad \boldsymbol{\rho} = \sum_{r=1}^{N}\left(1 + \frac{(\boldsymbol{\alpha}_r\cdot\boldsymbol{\alpha}_r)}{2}\right)\hat{\boldsymbol{\alpha}}_r \, ,$$

  and $(\boldsymbol{\alpha}_r \cdot \hat{\boldsymbol{\alpha}}_s) = \delta_{r,s}$. The corresponding central charge is

$$c = N + 12(\boldsymbol{\rho}\cdot\boldsymbol{\rho}) \, .$$

- For $N=1$ we have a current

$$T(\varphi) = -\frac{1}{2}(\partial\varphi)^2 + \left(\frac{1}{\alpha} + \frac{\alpha}{2}\right)\partial^2\varphi \, .$$

  The same algebra can be defined through the dual screening charge $\oint e^{\alpha^\vee \varphi} dz$ with $\alpha^\vee = \frac{2}{\alpha}$.

Bosonic and fermionic roots

- Depiction of bosonic roots

$$\bigcirc \text{ -- bosonic root: } (\boldsymbol{\alpha}_r \cdot \boldsymbol{\alpha}_r) = \text{generic}$$

- If the current W_s satisfies commutativity condition it should be of a special form

$$W_s = W_s\Big(T(\varphi_{\|}), \boldsymbol{\varphi}_{\perp}\Big)\,,$$

where

$$\varphi_{\|} \overset{\text{def}}{=} \frac{(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi})}{(\boldsymbol{\alpha}_r \cdot \boldsymbol{\alpha}_r)^{\frac{1}{2}}}, \quad \boldsymbol{\varphi}_{\perp} \overset{\text{def}}{=} \boldsymbol{\varphi} - \frac{(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi})}{(\boldsymbol{\alpha}_r \cdot \boldsymbol{\alpha}_r)}\boldsymbol{\alpha}_r\,,$$

and $T(\varphi_{\|})$ is given by $W_2(z)$ with $\alpha = (\boldsymbol{\alpha}_r \cdot \boldsymbol{\alpha}_r)^{\frac{1}{2}}$.

- Depiction of fermionic roots

$$\otimes \text{ -- fermionic root: } (\boldsymbol{\alpha}_r \cdot \boldsymbol{\alpha}_r) = -1$$

- In the coordinates defined above it corresponds to the complex fermion. The communant of the corresponding screening charge $\oint e^{-i\varphi_{\|}(z)}dz$ consists of all $w_s = \psi^{+}\partial^{s-1}\psi$, $s = 2, 3, \ldots$
- Among these currents only w_2 and w_3 are independent. Therefore

$$W_s = W_s\Big(w_2(\varphi_{\|}), w_3(\varphi_{\|}), \boldsymbol{\varphi}_{\perp}\Big).$$

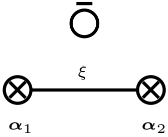
Properties of the systems with bosonic/fermionic roots

- **Bosonic root duality:** the bosonic roots always appear in pairs

$$\boldsymbol{\alpha} \quad \text{and} \quad \boldsymbol{\alpha}^{\vee} = \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})} \,.$$

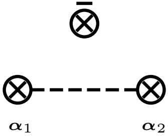
- **Dressed/sigma-model bosonic screening:** $(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2) = \xi$ is arbitrary

$$\mathcal{S}_B = \oint (\boldsymbol{\alpha}_1 \cdot \partial\boldsymbol{\varphi}) e^{(\boldsymbol{\beta}_{12}\cdot\boldsymbol{\varphi})} dz, \quad \text{where} \quad \boldsymbol{\beta}_{12} = \frac{2(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)^2}$$



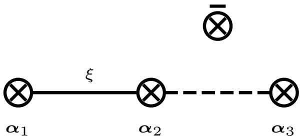
- **Dressed/sigma-model fermionic screening:** $(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2) = -1$

$$\mathcal{S}_F = \oint (\boldsymbol{\alpha}_1 \cdot \partial\boldsymbol{\varphi}) e^{(\boldsymbol{\beta}_{12}\cdot\boldsymbol{\varphi})} dz, \quad \text{where} \quad \boldsymbol{\beta}_{12} = \nu\boldsymbol{\alpha}_1 - (1+\nu)\boldsymbol{\alpha}_2$$



Dressed/sigma-model fermionic screening

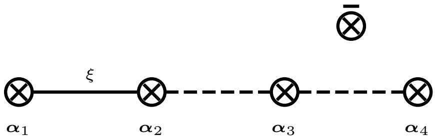
- The parameter ν cannot be fixed if only the two roots α_1 and α_2 are present.
- One way to fix the parameter ν is to embed in larger diagram. For example, consider the diagram



Then the parameter ν in the vector $\boldsymbol{\beta}_{23}$ is fixed from the condition

$$(\boldsymbol{\beta}_{23} \cdot \boldsymbol{\alpha}_1) = -1 \quad \Longrightarrow \quad \nu = -\frac{1}{\xi}.$$

- Another case also important for us is



Then the parameter ν in the vector $\boldsymbol{\beta}_{34}$ is fixed from the condition

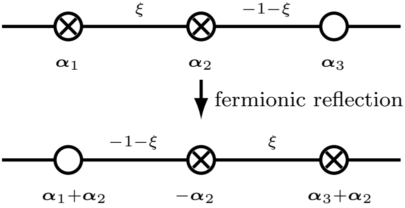
$$(\boldsymbol{\beta}_{34} \cdot \boldsymbol{\alpha}_2) = 1 - \xi \quad \Longrightarrow \quad \nu = \xi - 1.$$

Fermionic reflection

- There is another transformation, which involves given screening and neighbouring ones (Litvinov, Spodyneiko'16).
- This transormation is based on the Coulomb integral identities (Baseilhac, Fateev'99).
- If we have a CFT, defined by a set of screenings $\mathcal{S}_j = \oint e^{(\boldsymbol{\alpha}_j, \varphi(z))} dz$, then the same CFT is defined by a set of screenings $\tilde{\mathcal{S}}_j = \oint e^{(\tilde{\boldsymbol{\alpha}}_j, \varphi(z))} dz$ with

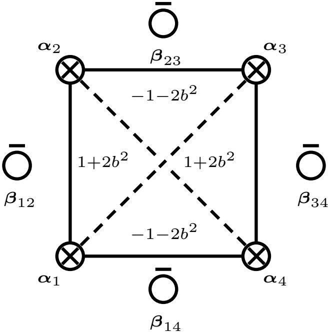
$$\tilde{\boldsymbol{\alpha}}_j = \begin{cases} -\boldsymbol{\alpha}_j \quad \text{if} \quad j = r\,, \\ \boldsymbol{\alpha}_j + \boldsymbol{\alpha}_r \quad \text{if} \quad (\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_r) \neq 0\,, \\ \boldsymbol{\alpha}_j \quad \text{otherwise} \end{cases}$$

  for the fermionic reflection with respect to the screening $\boldsymbol{\alpha}_r$.
- This operation can be illustrated with an example



Deformed $O(3)$ sigma model

- We want to check whether the metric is consistent with the screening charges corresponding to the η- and λ-deformed $O(3)$ sigma model (Fateev et al.'93).
- Let us recall that the theory in question may be determined by the following set of fermionic screenings

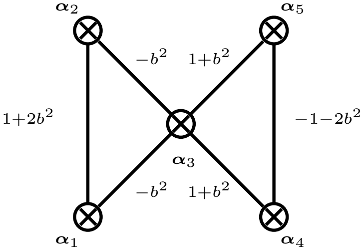


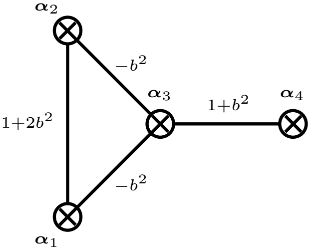
- By utilizing Cartesian coordinates as in (Litvinov, Spodyneiko'18) we can parametrize the fermionic screening lengths as follows

$$\boldsymbol{\alpha}_1 = bE_1 + i\beta e_1 \ , \quad \boldsymbol{\alpha}_2 = bE_1 - i\beta e_1 \ ,$$
$$\boldsymbol{\alpha}_3 = -bE_1 + i\beta e_1 \ , \quad \boldsymbol{\alpha}_4 = -bE_1 - i\beta e_1 \ .$$

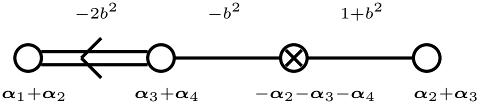
Deformed $O(5)$ sigma model

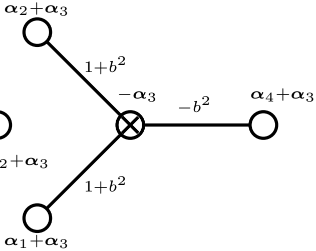
- Screening picture and corresponding underlying CFT $\frac{\widehat{\mathfrak{so}}(5)_{-b^2-3}}{\widehat{\mathfrak{so}}(4)_{-b^2-3}}$ with the central charge $c = 4 + \frac{30}{b^2} - \frac{12}{1+b^2}$ lead to the following diagrams





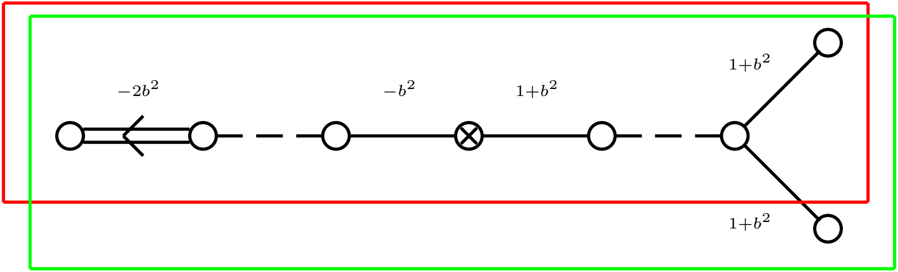
- Different applications of fermionic reflections lead to



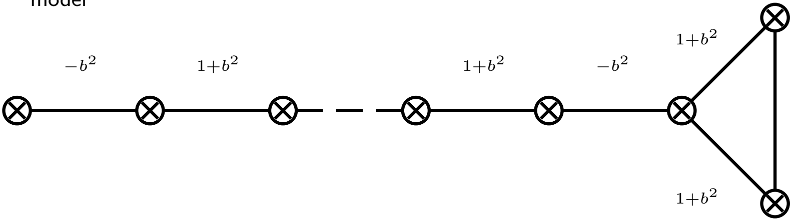


Deformed $O(N)$ model

- Therefore, these two representations above can be encoded in the following picture consisting of N screenings

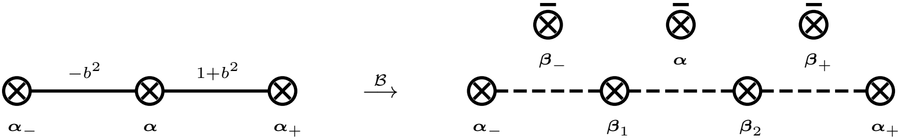


- Application of fermionic reflections in both cases leads to the CFT, integrable deformation of which leads to the set of screenings describing the $O(N)$ sigma model



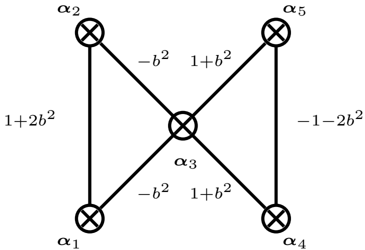
Blow-up transformation

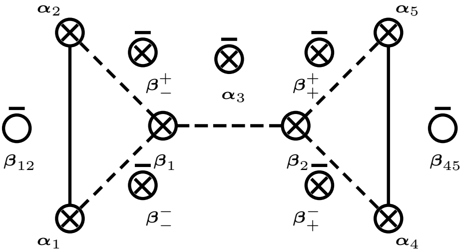
- Now we describe transformation $\mathcal{B}$ of the root system, we call it *blow-up*, which acts as
$$O(N) \to OSP(N|2) \ ,$$
or more generally as
$$OSP(N|2m) \to OSP(N|2m+2) \ .$$
It can be applied to both conformal diagram and its affine counterpart.
- It acts on any root except $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_{2n}$ and $\boldsymbol{\alpha}_{2n+1}$ and produces two fermionic roots out of one. On fermionic root $\boldsymbol{\alpha}$ it acts as follows
$$\boldsymbol{\alpha} = -b\boldsymbol{E} + i\beta\boldsymbol{e} \xrightarrow{\mathcal{B}} \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2\} = \left\{-\frac{1}{b}\boldsymbol{E} + \frac{i\beta}{b}\boldsymbol{\epsilon}, \frac{ib}{\beta}\boldsymbol{\epsilon} - \frac{i}{\beta}\boldsymbol{e}\right\} \ ,$$
where $\boldsymbol{\epsilon}$ is a new basis vector.
- Altogether this can be shown as follows



Screening charges for the deformed $OSp(5|2)$ sigma model

- Consider the simplest case of $OSP(5|2)$ affine diagram. According to our rule it is obtained from $O(5)$ diagram by blowing up the root $\boldsymbol{\alpha}_3$



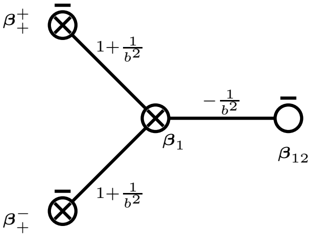


- The vectors $\boldsymbol{\alpha}_r$ can be parameterized as follows ($\beta = \sqrt{1+b^2}$)

$$\boldsymbol{\alpha}_1 = b\boldsymbol{E}_1 + i\beta\boldsymbol{e}_1\ , \quad \boldsymbol{\alpha}_2 = b\boldsymbol{E}_1 - i\beta\boldsymbol{e}_1\ , \quad \boldsymbol{\alpha}_3 = -b\boldsymbol{E}_1 + i\beta\boldsymbol{e}_2\ ,$$
$$\boldsymbol{\alpha}_4 = b\boldsymbol{E}_2 - i\beta\boldsymbol{e}_2\ , \quad \boldsymbol{\alpha}_5 = -b\boldsymbol{E}_2 - i\beta\boldsymbol{e}_2\ ,$$
$$\boldsymbol{\beta}_1 = -\frac{1}{b}\boldsymbol{E}_1 + \frac{i\beta}{b}\boldsymbol{\epsilon}\ , \quad \boldsymbol{\beta}_2 = \frac{ib}{\beta}\boldsymbol{\epsilon} - \frac{i}{\beta}\boldsymbol{e}_2\ , \quad \boldsymbol{\beta}_-^{\pm} = \pm\frac{i}{\beta}\boldsymbol{e}_1 - \frac{ib}{\beta}\boldsymbol{\epsilon}\ ,$$
$$\boldsymbol{\beta}_+^{\pm} = \pm\frac{1}{b}\boldsymbol{E}_2 - \frac{i\beta}{b}\boldsymbol{\epsilon}\ , \quad \boldsymbol{\beta}_{12} = \frac{1}{b}\boldsymbol{E}_1\ , \quad \boldsymbol{\beta}_{45} = \frac{i}{\beta}\boldsymbol{e}_2\ .$$

Metric for the deformed $OSp(5|2)$ sigma model

- By taking the dual screenings we obtain the following system, which includes the dressed screenings



- By choosing $z = x^1 - ix^2$ ($\bar{z} = x^1 + ix^2$) and then conducting Wick rotation $x^2 = ix^0$, we obtain the action in Minkowski signature

$$\mathcal{L} = \frac{1}{8\pi}\left(\sum_{i=1}^{2}(\partial_+\Phi_i)(\partial_-\Phi_i) + \sum_{j=1}^{3}(\partial_+\phi_j)(\partial_-\phi_j)\right) +$$

$$+\Lambda_1 e^{-\frac{i\beta}{b}\phi_3}\left(\partial_+\left(b\Phi_2 + i\beta\phi_2\right)\partial_-\left(b\Phi_2 - i\beta\phi_2\right)e^{-\frac{\Phi_2}{b}} +\right.$$

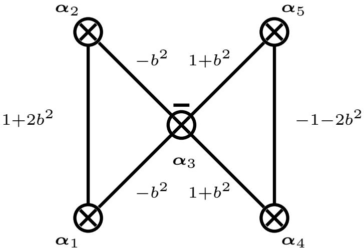
$$\left.+\partial_+\left(b\Phi_2 - i\beta\phi_2\right)\partial_-\left(b\Phi_2 + i\beta\phi_2\right)e^{\frac{\Phi_2}{b}}\right) + \Lambda_2 e^{-\frac{\Phi_1}{b}+\frac{i\beta}{b}\phi_3} +$$

$$+\Lambda_3\partial_+\left(b\Phi_1 + i\beta\phi_1\right)\partial_-\left(b\Phi_1 - i\beta\phi_1\right)e^{\frac{\Phi_1}{b}} + \frac{\pi b^2}{\beta^2}\Lambda_1\Lambda_2 e^{\frac{\Phi_1}{b}}\times$$

$$\times\left(\partial_+\left(b\Phi_2 + i\beta\phi_2\right)\partial_-\left(b\Phi_2 - i\beta\phi_2\right)e^{-\frac{\Phi_2}{b}} + \partial_+\left(b\Phi_2 - i\beta\phi_2\right)\partial_-\left(b\Phi_2 + i\beta\phi_2\right)e^{\frac{\Phi_2}{b}}\right) + \ldots\ ,$$

Screening charges in the $b \to 0$ limit

- By taking the subsystem of screenings, which are regular in the limit $b \to 0$



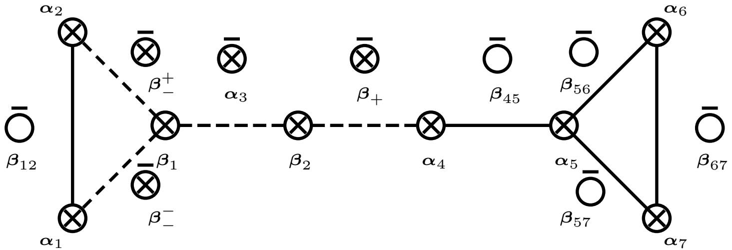
- We are able to write the lagrangian of the dual model

$$\mathcal{L} = \frac{1}{8\pi}\left(\sum_{i=1}^{2}(\partial\Phi_i)(\bar{\partial}\Phi_i) + \sum_{j=1}^{3}(\partial\varphi_j)(\bar{\partial}\varphi_j)\right) + 2\Lambda_1 e^{b\Phi_1}\cos\beta\varphi_1 +$$
$$+ \Lambda_2\partial(\Phi_1 - i\beta\varphi_3)\bar{\partial}(\Phi_1 + i\beta\varphi_3)e^{-b\Phi_1 + i\beta\varphi_2} +$$
$$+ \Lambda_3\left(e^{-b\Phi_2 - i\beta\varphi_2} + e^{b\Phi_2 - i\beta\varphi_2}\right) + (\text{counterterms})$$

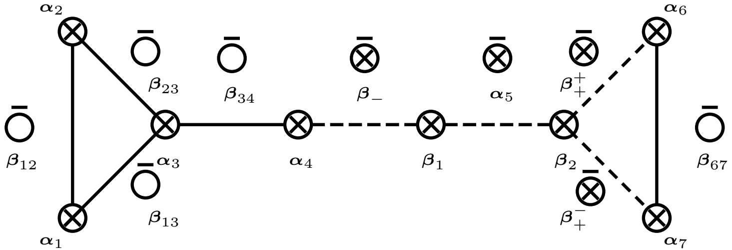
- This action appears to have only finite number of counterterms!

Deformed $OSp(7|2)$ sigma model

- There exist two integrable deformations of $OSp(7|2)$ sigma models, first of them is described by

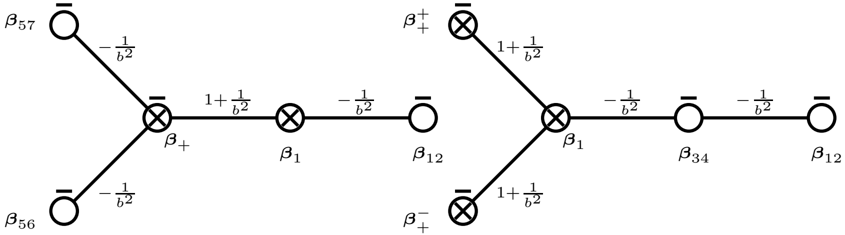


- The second one is described by the screenings

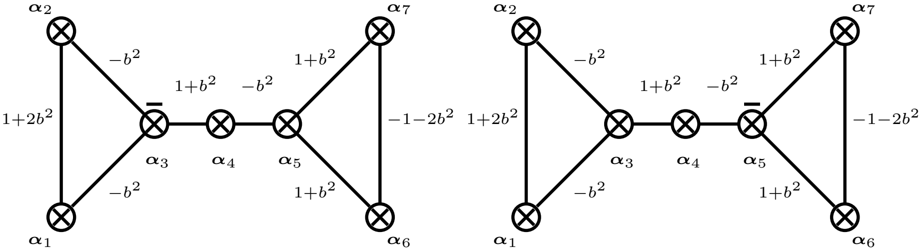


Metric and $b \to 0$ limit for the $OSp(7|2)$ sigma model

- Metric of the both deformations of $OSp(7|2)$ sigma model

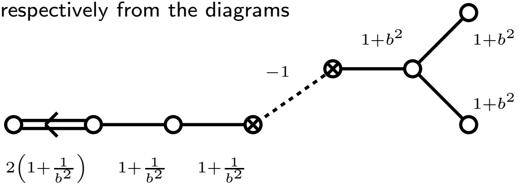


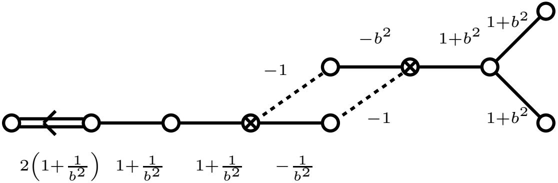
- Respectively in the $b \to 0$ limit we obtain the following screening charges



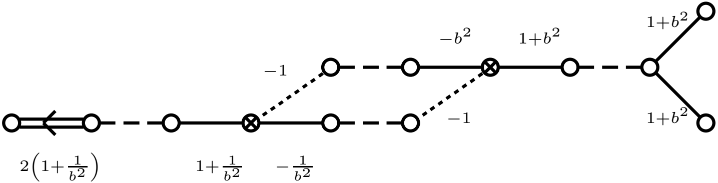
Set of screenings for general $OSp(N|2m)$ sigma model

- In the case of $OSp(7|2)$ and $OSp(7|4)$ we are able to obtain the underlying CFT respectively from the diagrams





- Based on the information above, we can put forward the hypothesis for the structure of the screening for general N and m



Conclusions

Results obtained:

- Presented a systematic way to generate the screening charges picture for deformed $O(N)$ sigma models.
- The system of screening charges, which determines the integrable structure of the $OSp(N|2)$ sigma model, was built.
- By using it we demonstrated how to restore the sigma model action in the deep UV in the cases of $OSp(5|2)$ and $OSp(7|2)$.
- Utilized our system of screenings to write the dual model with the Toda type interactions in the cases of $OSp(5|2)$ and $OSp(7|2)$.
- Put forward a hypothesis on the method to build the set of screening charges for general deformed $OSp(N|2m)$ sigma model.

Future goals:

- Find the system of screening charges for a wider class of integrable sigma models.
- The next interesting step would be to try to adapt the dual description for the sigma models with the non-compact target space (Basso, Zhong'18).
- Include reparametrization invariance into the dual description.

Thanks for your attention!