The Revealed Preference Lattice and The Lattice of Stable Matchings

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HSE

October 28, 2024

Representation of Individuals/Agents
in
Many to Many matching markets

Definitions

Consider an individual described by a CHOICE FN C on a universal set U:

$$C(T) \subset T \quad \forall T \subset U$$

and call $Z \equiv$ Image of C the CONSUMPTION SET.

The REVEALED PREFERENCE BINARY RELATION on Z is defined as:

$$S \succ S' \iff C(S \cup S') = S \quad \text{for } S, S' \in Z.$$

C is **path independent PI** if

$$C(T \cup T') = C(C(T) \cup T') \quad \text{for } T, T' \in U.$$

Definitions

C satisfies SUBSTITUTABILITY if:

$$T' \subset T \text{ and } a \in C(T) \cap T' \implies a \in C(T').$$

C satisfies CONSISTENCY if:

$$T' \subset T \text{ and } C(T) \subset T' \implies C(T) = C(T').$$

Aizerman Malishevsky: $PI \iff$ SUBST & CONS.

Theorem

The revealed preference relation $\succ$ of a path independent choice function on $Z \equiv$ Image of C is a lattice, where:

$$S \vee S' = C(S \cup S')$$

$$S \wedge S' = C(\overline{S} \cap \overline{S'})$$

and $\overline{S}$ is the largest superset of S such that $C(\overline{S}) = S$.

Proof

1. $\succ$ is a partial order:

$$S_1 \succ S_2 \text{ and } S_2 \succ S_3 \implies C(S_1 \cup S_3) = C(C(S_1 \cup S_2) \cup S_3)$$
$$= C(S_1 \cup C(S_2 \cup S_3)) = C(S_1 \cup S_2) = S_1$$

2. If $C(S_1 \cup S_2) \equiv S^*$ and $S^{**} \succ S_1$, $S^{**} \succ S_2$, then:

$$C(S^{**} \cup S^*) = C(S^{**} \cup C(S_1 \cup S_2))$$
$$= C(S^{**} \cup S_1 \cup S_2) = S^{**}$$

Thus, $(Z, \succ)$ is a lattice.

Proof (continued)

3. $S \leftrightarrow \overline{S}$ is one-to-one:

$$C(S') = S \text{ and } C(S'') = S \implies C(S' \cup S'' \cup S) = S$$

4. $\overline{S} \cap \overline{S'}$ is a closed set:

$$\implies a \notin \overline{S} \cap \overline{S'} \implies a \notin \overline{S} \text{ wlog}$$
$$\text{i.e., } a \in C(a \cup S) = C(a \cup \overline{S}) \implies a \in C(a \cup \overline{S} \cup \hat{S}')$$

by SUBST.

Thus, (Z, C) is a lattice.

5. $Z \leftrightarrow \overline{Z}$ is a lattice isomorphism:

$$\overline{S} \subset \overline{S'} \implies C(S \cup S') = C(C(\overline{S}) \cup C(\overline{S'}))$$
$$= C(\overline{S'}) = S' \implies S' \succ S$$

Therefore, $S \wedge S' = C(\overline{S} \cup \overline{S'})$.

Many to Many Matching Market

$$(F, W; (C_f), (C_w))$$

A matching is any subset of $F \times W$.

A matching μ is STABLE if there is no BLOCKING PAIR (f, w), i.e., no $(f, w) \notin \mu$ such that:

$$w \in C_f(\mu(f) \cup w) \text{ and } f \in C_w(\mu(w) \cup f)$$

(in words: no $(f, w) \notin \mu$ such that f likes w at μ and w likes f at μ.)

Many to Many Matching Market

EXISTENCE: When C_f, C_w are path independent, the Gale-Shapley Deferred Acceptance Procedure gives a stable matching.

Group Preferences

A matching μ is $\boldsymbol{F}$**-preferred** to μ', $\mu \geq_F \mu'$,
if every f "prefers" $\mu(f)$ to $\mu'(f)$ or $\mu(f) = \mu'(f)$,

that is, when C_f, C_w are path independent, if:

$$C_f(\mu(f) \cup \mu'(f)) = \mu(f) \quad \text{for every } f \in F.$$

The One to One Matching Market

Theorem (Conway-Knuth). The set of all stable matchings is a (distributive) lattice under $\geq_F$ or $\geq_W$, where

$$\mu \bigvee_F \mu' = (\mu(f) \vee_f \mu'(f)) = \mu^F$$

$$\mu \bigwedge_F \mu' = (\mu(f) \wedge_f \mu'(f)) = \mu_F$$

and $\mu^F = \mu_W$.

Back to Many to Many

Assume C_f, C_w are **path independent** $(F, W, (\succ_f), (\succ_w))$. Assume in addition C_f, C_w are size monotone i.e., $|C(T)| \leq |C(T')|$ if $T \subset T'$.

Theorem: The set of all stable matchings is a distributive lattice under $\geq_F$ (or $\geq_W$) where

$$\mu \bigvee_F \mu' = C_f(\mu(f) \cup \mu'(f)) = (\mu(f) \vee_f \mu'(f)) = \mu^F$$

$$\mu \bigwedge_F \mu' = C_f(\overline{\mu(f)} \cap \overline{\mu'(f)}) = (\mu(f) \wedge_f \mu'(f)) = \mu_F$$

and $\mu^F = \mu_W$.

Meet Lemma

$$S_1 \wedge S_2 \supset S_1 \cap \overline{S_2}$$

Proof:

$$\overline{S_1} \cap \overline{S_2} \subset \overline{S_1}$$
$$C(\overline{S_1}) = S_1$$

by SUBST: $C(\overline{S_1}) \cap (\overline{S_1} \cap \overline{S_2}) \subset C(\overline{S_1} \cap \overline{S_2})$

$\Rightarrow \quad S_1 \cap \overline{S_2} \subset S_1 \wedge S_2$

Proof

Take any $\mu, \mu' \in \Sigma$.

- $\mu^F \subset \mu_W$
  Say $fw \in \mu^F$.
  If $w \in \mu(f) \cap \mu'(f)$ then $f \in \mu(w) \cap \mu'(w)$.
  If $w \in \mu(f) - \mu'(f)$ then by SUBST, f likes w at μ'.
  So by STABILITY, w doesn't like f at μ.
  So $f \in \mu'(w)$.

  So, $f \in \mu(w) \cap \mu'(w)$
  $f \in \mu(w) \vee_w \mu'(w)$
  $fw \in \mu_W$
  By Meet Lemma.

- $|\mu^F| \leq |\mu_W| \leq |\mu^W| \leq |\mu_F| \leq |\mu^F|$
  By the • above and by size monotonicity,
  so $\mu^F = \mu_W$.

Proof (continued)

- μ^F is individually rational.
- μ^F is stable:

  say $fw \notin \mu^F = \mu_W$ and f likes w at μ^F.
  By SUBST, f likes w at μ and at μ'.
  By STABILITY, w does not like f at μ or μ',
  i.e., $f \in \overline{\mu(w)} \cap \overline{\mu'(w)}$.

$$C(\mu_W \cup f) = C\left(C_W\left(\overline{\mu(w)} \cap \overline{\mu'(w)} \cup f\right)\right) =$$

$$C\left(\overline{\mu(w)} \cap \overline{\mu'(w)}\right) = \mu_W(w)$$

  i.e., w does not like f at μ^F.

Remarks

- Preference Ordering $\rightarrow$ Choice Fn (PI)
  Choice Fn $\rightarrow$ Revealed Preference Relation (Lattice)
- Rationalizing Utility Fn
  Utility Fn of 2^U maximization of which gives the Choice Fn
- Birkhoff Representation Theorem for Distributive Lattices
  Rotations – generate the entire lattice
- Alkan Yildiz
  Faenza *et al.*