Polyhedral products, graph products and p-central series

joint with Temurbek Rahmatullaev

Taras Panov

Moscow State University & Higher School of Economics

Workshop on Polyhedral Products
Fields Institute, Toronto, July 29–August 2, 2024

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\varnothing \in \mathcal{K}$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \dots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ over $\mathcal{K}$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Big) \subset \prod_{i=1}^{m} X_i.$$

Notation: $(X, A)^{\mathcal{K}} = (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;
$\boldsymbol{X}^{\mathcal{K}} = (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

Category of faces $\mathrm{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\begin{aligned}\mathcal{T}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})\colon \mathrm{CAT}(\mathcal{K}) &\longrightarrow \mathrm{TOP},\\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I,\end{aligned}$$

which maps the morphism $I \subset J$ of $\mathrm{CAT}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

Then

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \operatorname{colim}^{\mathrm{TOP}} \mathcal{T}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname{colim}^{\mathrm{TOP}}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

C a symmetric monoidal category with finite colimits.

$\boldsymbol{X} = (X_1, \ldots, X_m)$ a sequence of objects in C. Set

$$\boldsymbol{X}^I = \prod_{i \in I} X_i,$$

where the product is taken with respect to the monoidal structure.

Consider the diagram

$$\mathcal{C}_{\mathcal{K}}(\boldsymbol{X})\colon \ \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{C}, \qquad I \longmapsto \boldsymbol{X}^I,$$

taking a morphism $I \subset J$ to the canonical morphism $\boldsymbol{X}^I \to \boldsymbol{X}^J$.

The polyhedral product

$$\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{c}} \mathcal{C}_{\mathcal{K}}(\boldsymbol{X}) = \operatorname{colim}^{\mathrm{c}}_{I \in \mathcal{K}} \boldsymbol{X}^I.$$

$\partial\Delta_{[m]}$: boundary of simplex on $[m]$.

The polyhedral product $\boldsymbol{X}^{\partial\Delta_{[m]}}$ of pointed topological spaces $X_1, \ldots, X_m$ is known as their fat wedge. When $m = 2$, it is the wedge $X_1 \vee X_2$.

Definition

A symmetric monoidal category C with finite colimits is fat if the canonical morphism

$$\boldsymbol{X}^{\partial\Delta_{[m]}} \to \boldsymbol{X}^{\Delta_{[m]}} = \prod_{i=1}^{m} X_i$$

is an isomorphism for any objects $X_1, \ldots, X_m$ and $m \geq 3$.

Examples of fat categories:

- GRP: groups (the monoidal product is cartesian, the coproduct is the free product);
- TMN: topological monoids (the monoidal product is cartesian, the coproduct is the free product);
- TGP: topological groups (a full subcategory of TMN);
- ASS: associative algebras with unit over a commutative ring R (the monoidal product is the tensor product, the coproduct is the free product);
- LIE: Lie algebras over R (the monoidal product is the direct sum, the coproduct is the free product).

In each of these categories, a set of pairwise commuting elements of a group (algebra) generates a commutative subgroup (subalgebra).

On the other hand, the category TOP is clearly not fat.

Proposition

In a fat category $\mathcal{C}$, the morphism $\boldsymbol{X}^{\mathcal{K}^1} \to \boldsymbol{X}^{\mathcal{K}}$ is an isomorphism for any simplicial complex $\mathcal{K}$ and sequence of objects $\boldsymbol{X} = (X_1, \ldots, X_m)$.

Γ a simple graph.

Complete subgraphs in Γ form a flag simplicial complex $\mathcal{K}(\Gamma)$, the clique complex of Γ.

Every flag complex $\mathcal{K}$ is the clique complex of its 1-skeleton $\mathcal{K}^1$.

For a sequence $\boldsymbol{X} = (X_1, \ldots, X_m)$ of objects in $\mathcal{C}$ and a simple graph Γ on m vertices, the graph product $\boldsymbol{X}^{\Gamma}$ is the polyhedral product $\boldsymbol{X}^{\mathcal{K}(\Gamma)}$.

If $\mathcal{C}$ is fat, then $\boldsymbol{X}^{\Gamma}$ is canonically isomorphic to $\boldsymbol{X}^{\mathcal{K}}$ for any $\mathcal{K}$ with $\mathcal{K}^1 = \Gamma$.

Example (graph product of groups)

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m (topological) groups.

Since the category TGP is fat, we have

$$\boldsymbol{G}^{\mathcal{K}^1} = \boldsymbol{G}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{TGP}}_{I\in\mathcal{K}} \boldsymbol{G}^I.$$

Explicitly,

$$\boldsymbol{G}^{\mathcal{K}} = \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \text{ for } g_i \in G_i,\ g_j \in G_j,\ \{i,j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ denotes the free product of the groups G_k.

There are canonical injective homomorphisms $\boldsymbol{G}^I \to \boldsymbol{G}^{\mathcal{K}}$ for $I \in \mathcal{K}$.

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

BG the classifying space for G, $EG \to BG$ the universal G-bundle.

$B\boldsymbol{G}^I = \prod_{i \in I} BG_i$.

$(B\boldsymbol{G})^{\mathcal{K}}$ the polyhedral product corresponding to the sequence of pairs $(B\boldsymbol{G}, pt) = \{(BG_1, pt), \ldots, (BG_m, pt)\}$ in TOP.

$(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ for $(E\boldsymbol{G}, \boldsymbol{G}) = \{(EG_1, G_1), \ldots, (EG_m, G_m)\}$.
Here each G_i is included in EG_i as the fibre of $EG_i \to BG_i$ over pt.

There is a homotopy fibration of polyhedral products

$$(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \longrightarrow (B\boldsymbol{G})^{\mathcal{K}} \longrightarrow \prod_{i=1}^{m} BG_i.$$

The classifying space functor $B\colon \mathrm{TGP} \to \mathrm{TOP}$ does not preserve colimits over small categories in general, but it does preserve *homotopy colimits* under appropriate model structures on TOP and TGP.

Theorem (P-Ray-Vogt)

There is a commutative square of natural maps

$$\begin{array}{ccc} \operatorname{hocolim}^{\mathrm{TOP}}_{I\in\mathcal{K}} B\boldsymbol{G}^I & \xrightarrow{\simeq} & B\operatorname{hocolim}^{\mathrm{TGP}}_{I\in\mathcal{K}} \boldsymbol{G}^I \\ \big\downarrow{\simeq} & & \big\downarrow \\ \operatorname{colim}^{\mathrm{TOP}}_{I\in\mathcal{K}} B\boldsymbol{G}^I & \longrightarrow & B\operatorname{colim}^{\mathrm{TGP}}_{I\in\mathcal{K}} \boldsymbol{G}^I, \end{array}$$

in TOP, where the top and left maps are homotopy equivalences, and the right map is a homotopy equivalence when $\mathcal{K}$ is a flag complex.

Corollary

$(B\boldsymbol{G})^{\mathcal{K}} \to B(\boldsymbol{G}^{\mathcal{K}})$ is a homotopy equivalence when $\mathcal{K}$ is flag.
That is, the classifying space functor preserves graph products.

Example

Let $\mathcal{K}$ be a flag complex in these examples.

1. Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = \mathbb{Z}^{\mathcal{K}} = F(a_1, \ldots, a_m) \big/ \big(a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}\big),$$

where $F(a_1, \ldots, a_m)$ is a free group on m generators. The classifying space of $RA_{\mathcal{K}}$ is the polyhedral product $(B\mathbb{Z})^{\mathcal{K}} = (S^1)^{\mathcal{K}}$, a subcomplex of the m-torus $T^m = (S^1)^m$. There is a homotopy fibration

$$(\mathbb{R}, \mathbb{Z})^{\mathcal{K}} \longrightarrow (S^1)^{\mathcal{K}} \longrightarrow (S^1)^m,$$

where all spaces are aspherical. Passing to the fundamental groups,

$$1 \longrightarrow RA'_{\mathcal{K}} \longrightarrow RA_{\mathcal{K}} \xrightarrow{Ab} \mathbb{Z}^m \longrightarrow 1,$$

where Ab is the abelianisation map and $RA'_{\mathcal{K}}$ is the commutator subgroup.

Example

2. Let $G_i = \mathbb{Z}_2$, a cyclic group of order 2, for $i = 1, \ldots, m$.
Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Coxeter group

$$RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}} = F(a_1, \ldots, a_m) \big/ \big(a_i^2 = 1,\ a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K} \big).$$

The classifying space of $RC_{\mathcal{K}}$ is $(B\mathbb{Z}_2)^{\mathcal{K}} = (\mathbb{R}P^\infty)^{\mathcal{K}}$.
The polyhedral product $(E\mathbb{Z}_2, \mathbb{Z}_2)^{\mathcal{K}}$ is homotopy equivalent to the real moment-angle complex $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$.There is a homotopy fibration

$$\mathcal{R}_{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^m,$$

where all spaces are aspherical. Passing to the fundamental groups,

$$1 \longrightarrow RC'_{\mathcal{K}} \longrightarrow RC_{\mathcal{K}} \xrightarrow{Ab} \mathbb{Z}_2^m \longrightarrow 1.$$

Example

3. Let $G_i = T^1$ (a circle group) for $i = 1, \dots, m$. The corresponding graph product $\boldsymbol{G}^{\mathcal{K}}$ is the colimit of tori T^I, known as the circulation group:

$$Cir_{\mathcal{K}} = T^{\mathcal{K}} = \operatorname{colim}^{\mathrm{TGP}}_{I \in \mathcal{K}} T^I.$$

Its classifying space is $(BT^1)^{\mathcal{K}} = (\mathbb{C}P^\infty)^{\mathcal{K}}$, the Davis–Januszkiewicz space. The polyhedral product $(ET^1, T^1)^{\mathcal{K}}$ is homotopy equivalent to the moment-angle complex $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$. There is a homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m.$$

Applying the Moore loop space functor $\Omega\colon \mathrm{TOP} \to \mathrm{TMN}$ we obtain a commutative diagram in $Ho(\mathrm{TMN})$

$$\begin{array}{ccccccccc}
 & & \Omega\mathcal{Z}_{\mathcal{K}} & \longrightarrow & \Omega(\mathbb{C}P^\infty)^{\mathcal{K}} & \longrightarrow & T^m & & \\
 & & \downarrow\simeq & & \downarrow\simeq & & \| & & \\
1 & \longrightarrow & Cir'_{\mathcal{K}} & \longrightarrow & Cir_{\mathcal{K}} & \xrightarrow{Ab} & T^m & \longrightarrow & 1
\end{array}$$

Example

4. More generally, let G_i be the Moore loop monoid ΩX_i of a simply connected space X_i, $i = 1, \dots, m$. There is an equivalence in $Ho(\mathrm{TMN})$

$$\Omega(\boldsymbol{X}^{\mathcal{K}}) \simeq (\Omega \boldsymbol{X})^{\mathcal{K}} = \mathrm{colim}^{\mathrm{TMN}}_{I \in \mathcal{K}} (\Omega \boldsymbol{X})^I$$

for flag $\mathcal{K}$. The Moore loop functor preserves graph products up to htopy.

R a commutative ring R with unit.
The loop homology $H_*(\Omega X; R)$ of a simply connected space X is an associative noncommutative algebra with respect to the Pontryagin product.
Get a functor $H_*\Omega\colon \mathrm{TOP} \to \mathrm{ASS}$.

When R is a field or $H_*(\Omega X; R)$ is free over R, there is a cocommutative coproduct $H_*(\Omega X; R) \to H_*(\Omega X; R) \otimes H_*(\Omega X; R)$ induced by the diagonal map $\Delta\colon \Omega X \to \Omega X \times \Omega X$, which makes $H_*(\Omega X; R)$ into a cocommutative Hopf algebra.

Given a sequence $\boldsymbol{A} = (A_1, \ldots, A_m)$ of associative algebras, the polyhedral product

$$\boldsymbol{A}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{ASS}}_{I \in \mathcal{K}} \Big(\bigotimes_{i \in I} A_i\Big)$$

is also the graph product, since the category ASS is fat.

Theorem (Dobrinskaya, Cai)

Let $\mathcal{K}$ be a flag complex, $\boldsymbol{X} = (X_1, \ldots, X_m)$ a sequence of simply connected spaces, and R a field. There are isomorphisms

$$H_*\big(\Omega(\boldsymbol{X}^{\mathcal{K}}); R\big) \cong \big(H_*(\Omega \boldsymbol{X}; R)\big)^{\mathcal{K}}$$

$$\cong \mathop{\bigstar}_{k=1}^{m} H_*(\Omega X_i) \big/ \big([x_i, x_j] = 0 \ \text{ for } x_i \in H_*(\Omega X_i),\ x_j \in H_*(\Omega X_j),\ \{i,j\} \in \mathcal{K}\big)$$

of graded R-algebras, where $[x_i, x_j] = x_i x_j - (-1)^{|x_i||x_j|} x_j x_i$.
That is, the loop homology functor $H_*\Omega \colon \mathrm{TOP} \to \mathrm{ASS}$ with field coefficients preserves graph products.

The cases $X_i = S^{2p+1}$ and $X_i = \mathbb{C}P^\infty$ are of particular importance.

We have $H_*(\Omega S^{2p+1}; R) = R[v]$, $|v| = 2p$.

For flag $\mathcal{K}$, the graph product algebra $H_*(\Omega(S^{2p+1})^{\mathcal{K}}; R)$ is the algebraic counterpart of the right-angled Artin group:

Theorem (Bubenik–Gold)

For any flag complex $\mathcal{K}$ and $p \geq 1$, there are isomorphisms

$$\begin{aligned} H_*\bigl(\Omega(S^{2p+1})^{\mathcal{K}}; R\bigr) &\cong R[v]^{\mathcal{K}} = \operatorname{colim}^{\mathrm{ASS}}_{I \in \mathcal{K}} R[v_i \colon i \in I] \\ &\cong T(v_1, \ldots, v_m)/(v_i v_j - v_j v_i = 0 \text{ for } \{i, j\} \in \mathcal{K}) \end{aligned}$$

of graded R-algebras, where $T(v_1, \ldots, v_m)$ is the free associative algebra, $R[v_i \colon i \in I]$ is the polynomial algebra and $|v_i| = 2p$.

Similarly, the algebra $H_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K}; R)$ is related to right-angled Coxeter groups. To elaborate on this, consider the exterior algebra

$$H_*(\Omega(\mathbb{C}P^\infty)^m; R) = H(T^m; R) = \Lambda[u_1, \ldots, u_m],$$

where $|u_i| = 1$.The homotopy fibration $\mathcal{Z}_\mathcal{K} \to (\mathbb{C}P^\infty)^\mathcal{K} \to (\mathbb{C}P^\infty)^m$ gives

$$R \longrightarrow H_*(\Omega\mathcal{Z}_\mathcal{K}; R) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K}; R) \xrightarrow{Ab} \Lambda[u_1, \ldots, u_m] \longrightarrow R.$$

$H_*(\Omega\mathcal{Z}_\mathcal{K}; R)$ is the commutator subalgebra of $H_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K}; R)$.

Theorem (P–Ray)

For any flag complex $\mathcal{K}$, there are isomorphisms

$$\begin{aligned} H_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K}; R) &\cong \Lambda[u]^\mathcal{K} = \operatorname{colim}^{\mathrm{ASS}}_{I\in\mathcal{K}} \Lambda[u_i \colon i \in I] \\ &\cong T(u_1, \ldots, u_m)/(u_i^2 = 0,\ u_iu_j + u_ju_i = 0 \text{ for } \{i,j\} \in \mathcal{K}) \end{aligned}$$

of graded R-algebras, where $\Lambda[u_i \colon i \in I]$ is the exterior algebra and $|u_i| = 1$.

Theorem (Grbić–P–Theriault–Wu)

Assume that $\mathcal{K}$ is flag. The commutator subalgebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is generated by $\sum_{I\subset[m]}\dim \widetilde{H}^0(\mathcal{K}_I)$ iterated commutators of the form

$$[u_j,u_i],\quad [u_{k_1},[u_j,u_i]],\quad \ldots,\quad [u_{k_1},[u_{k_2},\cdots[u_{k_{m-2}},[u_j,u_i]]\cdots]]$$

where $k_1<k_2<\cdots<k_p<j>i$, $k_s\ne i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal{K}_{\{k_1,\ldots,k_p,j,i\}}$. Furthermore, this multiplicative generating set is minimal, that is, the commutators above form a basis in the submodule of indecomposables in $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

Theorem (P–Veryovkin)

The commutator subgroup $RC'_\mathcal{K}=\pi_1(\mathcal{R}_\mathcal{K})=H_*(\Omega\mathcal{R}_\mathcal{K})$ has a minimal generator set consisting of $\sum_{J\subset[m]}\operatorname{rank}\widetilde{H}_0(\mathcal{K}_J)$ iterated commutators

$$(g_j,g_i),\quad (g_{k_1},(g_j,g_i)),\quad \ldots,\quad (g_{k_1},(g_{k_2},\cdots(g_{k_{m-2}},(g_j,g_i))\cdots)),$$

with the same condition on the indices as in the previous theorem.

Central series and associated Lie algebras

Given two subgroups H and W of G, denote by (H, W) the subgroup generated by all commutators (h, w) with $h \in H$ and $w \in W$.
In particular, $(G, G) = G'$ is the commutator subgroup.

A (descending) central series or central filtration on G is a sequence of subgroups $\Gamma(G) = \{\Gamma_k(G)\colon k \geq 1\}$ such that $\Gamma_1(G) = G$, $\Gamma_{k+1}(G) \subset \Gamma_k(G)$ and $(\Gamma_k(G), \Gamma_l(G)) \subset \Gamma_{k+\ell}(G)$ for any k, ℓ.

The fastest descending central series is the lower central series $\gamma(G)$, given by $\gamma_1(G) = G$ and $\gamma_k(G) = (\gamma_{k-1}(G), G)$ for $k \geq 2$.

The associated Lie algebra of a central series $\Gamma(G)$ is the associated graded abelian group

$$\mathrm{gr}_\bullet \Gamma(G) = \bigoplus_{k \geq 1} \Gamma_k(G)/\Gamma_{k+1}(G)$$

with the Lie bracket defined by $[\overline{g}_k, \overline{g}_\ell] = \overline{(g_k, g_\ell)}$, where $\overline{g}_k$ denotes the image of $g_k \in \Gamma_k(G)$ in the quotient group $\Gamma_k(G)/\Gamma_{k+1}(G)$.

By the classical result of Magnus, the Lie algebra associated with the lower central series of a free group $F(a_1, \ldots, a_m)$ is a free Lie algebra $FL(\bar{a}_1, \ldots, \bar{a}_m)$ over $\mathbb{Z}$. This generalises to right-angled Artin groups:

Theorem (Duchamp–Krob, Papadima–Suciu)

The Lie algebra associated with the lower central series of a right-angled Artin group $RA_{\mathcal{K}}$ is given by

$$\operatorname{gr}_\bullet \gamma(RA_{\mathcal{K}}) \cong FL(v_1, \ldots, v_m) \big/ \big([v_i, v_j] = 0 \textit{ for } \{i, j\} \in \mathcal{K}\big).$$

Furthermore, each component $\operatorname{gr}_k \gamma(RA_{\mathcal{K}})$ is a free abelian group.

The Lie algebra (over $\mathbb{Z}$) on the right hand side is known as the graph Lie algebra, or partially commutative Lie algebra. It is the graph product $\mathbb{Z}\langle v\rangle^{\mathcal{K}}$ of trivial Lie algebras $\mathbb{Z}\langle v\rangle = FL(v)$.

The universal enveloping functor $\mathcal{U}\colon \mathrm{LIE} \to \mathrm{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

Proposition

For a flag complex $\mathcal{K}$, there is an isomorphism

$$\mathcal{U}(\operatorname{gr}_\bullet \gamma(RA_\mathcal{K})) \cong H_*(\Omega(S^{2p+1})^\mathcal{K}; \mathbb{Z})$$

of associative $\mathbb{Z}$-algebras, where $p \geq 1$.

The lower central series of a right-angled Coxeter group $RC_\mathcal{K}$ is more subtle. One has $(\gamma_k(RC_\mathcal{K}))^2 \subset \gamma_{k+1}(RC_\mathcal{K})$, so $\operatorname{gr}_\bullet \gamma(RC_\mathcal{K})$ is a Lie algebra over $\mathbb{Z}_2$. There is an epimorphism

$$FL_{\mathbb{Z}_2}(v_1, \ldots, v_m)\big/\big([v_i, v_j] = 0 \text{ for } \{i,j\} \in \mathcal{K}\big) \to \operatorname{gr}_\bullet \gamma(RC_\mathcal{K})$$

of Lie algebras over $\mathbb{Z}_2$, where $FL_{\mathbb{Z}_2}(v_1, \ldots, v_m) = FL(v_1, \ldots, v_m) \otimes \mathbb{Z}_2$. This fails to be an isomorphism already when $\mathcal{K}$ is two disjoint points and $RC_\mathcal{K} = \mathbb{Z}_2 \star \mathbb{Z}_2$ [Veryovkin].

Restricted Lie algebras and p-central series

Let p be a prime and let R be a field of characteristic p.

A restricted Lie algebra over R, or shortly a p-Lie algebra, is a Lie algebra L over R equipped with a p-power operation $x \mapsto x^{[p]}$ satisfying

1. $[x, y^{[p]}] = [\ldots[[x, \underbrace{y], y], \ldots, y]}_{p}$ for $x, y \in L$;
2. $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for $x \in L$ and $\alpha \in R$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

   where $s_i(x,y)$ is defined by the expansion

   $$[\ldots[[x, \underbrace{\lambda x + y], \lambda x + y], \ldots, \lambda x + y]}_{p-1} = \sum_{i=1}^{p-1} i s_i(x,y)\lambda^{i-1}$$

   for $x, y \in L$ and $\lambda \in R$.

For example, if $p = 2$ then $(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x,y]$, and if $p = 3$ then $(x+y)^{[3]} = x^{[3]} + y^{[3]} + [[x,y],y] - [[x,y],x]$.

A *p-central series* on G is a central series $\Gamma^{[p]}(G) = \{\Gamma^{[p]}_k(G)\colon k \ge 1\}$ such that if $g \in \Gamma^{[p]}_k(G)$, then $g^p \in \Gamma^{[p]}_{pk}(G)$ for all k.

Each quotient $\Gamma^{[p]}_k(G)/\Gamma^{[p]}_{k+1}(G)$ is a $\mathbb{Z}_p$-module.

The *lower p-central series* is the fastest descending p-central series. It is given by $\gamma^{[p]}_1(G) = G$ and $\gamma^{[p]}_k(G) = \bigl(\gamma^{[p]}_{k-1}(G), G\bigr)\bigl(\gamma^{[p]}_{\lceil k/p\rceil}\bigr)^p$. Explicitly,

$$\gamma^{[p]}_k(G) = \prod_{mp^i \ge k} \bigl(\gamma_m(G)\bigr)^{p^i}.$$

The *associated p-Lie algebra* is the graded $\mathbb{Z}_p$-module

$$\mathrm{gr}_\bullet\, \Gamma^{[p]}(G) = \bigoplus_{k\ge 1} \Gamma^{[p]}_k(G)/\Gamma^{[p]}_{k+1}(G)$$

with the bracket and p-power defined by

$$[\overline{g}_k, \overline{g}_\ell] = \overline{(g_k, g_\ell)}, \quad (\overline{g}_k)^{[p]} = \overline{g^p_k}.$$

Recall $RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}}$. More generally, consider

$$\mathbb{Z}_p^{\mathcal{K}} = \operatorname{colim}^{\mathrm{GRP}}_{I\in\mathcal{K}} \mathbb{Z}_p^I = F(a_1,\ldots,a_m)\big/\big(a_i^p = 1,\ a_ia_j = a_ja_i \text{ for } \{i,j\}\in\mathcal{K}\big).$$

Theorem (P-Rahmatullaev)

For any simplicial complex $\mathcal{K}$ and prime p, there is an isomorphism

$$\operatorname{gr}_\bullet \gamma^{[p]}(\mathbb{Z}_p^{\mathcal{K}}) \cong FL_p(u_1,\ldots,u_m)\big/\big(u_i^{[p]} = 0,\ [u_i,u_j] = 0 \text{ for } \{i,j\}\in\mathcal{K}\big)$$

of p-Lie algebras over $\mathbb{Z}_p$.

The proof uses Quillen's isomorphism of graded R-algebras

$$\mathcal{U}_p\big(\operatorname{gr}_\bullet \gamma^{[p]}(G)\big)\otimes_{\mathbb{Z}} R \cong \operatorname{gr}_\bullet(R[G]),$$

where $\mathcal{U}_p(-)$ is the universal enveloping of a p-Lie algebra and $\operatorname{gr}_\bullet(R[G])$ is the graded ring associated with the filtration of the group algebra $R[G]$ by the powers of the augmentation ideal.

Proposition

The functor $\operatorname{gr}_\bullet \gamma^{[p]} \colon \mathrm{GRP} \to \mathrm{LIE}_p$ *preserves graph products of elementary abelian p-groups.*

Theorem

For any simplicial complex $\mathcal{K}$, *there is an isomorphism*

$$\operatorname{gr}_\bullet \gamma^{[2]}(RC_\mathcal{K}) \cong FL_2(u_1, \ldots, u_m) / \big(u_i^{[2]} = 0,\ [u_i, u_j] = 0 \text{ for } \{i,j\} \in \mathcal{K}\big)$$

of 2*-Lie algebras over* $\mathbb{Z}_2$.

Proposition

For a flag complex $\mathcal{K}$, *there is an isomorphism*

$$\mathcal{U}_2\big(\operatorname{gr}_\bullet \gamma^{[2]}(RC_\mathcal{K})\big) \cong H_*\big(\Omega(\mathbb{C}P^\infty)^\mathcal{K}; \mathbb{Z}_2\big)$$

of associative $\mathbb{Z}_2$*-algebras.*

References

[1] T. Panov, N. Ray and R. Vogt. *Colimits, Stanley-Reiner algebras, and loop spaces*, in: "Categorical Decomposition Techniques in Algebraic Topology" (G. Arone et al eds.), Progress in Mathematics, vol. 215, Birkhauser, Basel, 2004, pp. 261–291.

[2] J. Grbić, T. Panov, S. Theriault and J. Wu. *The homotopy types of moment-angle complexes for flag complexes.* Trans. of the Amer. Math. Soc. 368 (2016), no. 9, 6663–6682.

[3] T. Panov and Ya. Veryovkin. *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups*. Sbornik Math. 207 (2016), no. 11, 1582–1600.

[4] J. Grbić, M. Ilyasova, T. Panov and G. Simmons. *One-relator groups and algebras related to polyhedral products*. Proc. Roy. Soc. Edinburgh Sect. A 152 (2022), no. 1, 128–147.

[5] T. Panov and T. Rahmatullaev. *Polyhedral products, graph products and p-central series*. arXiv:2402.11556