Weak Belief and Permissibility

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Abstract. We characterize Permissibility as the behavioral implications of Cautious Rationality and Common Weak Belief of Cautious Rationality.

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1 Introduction

Permissibility (Brandenburger, [3]) is a solution concept for games in strategic form, based on a iterative procedure. A strategy survives a step of the iterative procedure if it is a lexicographic best reply to a Lexicographic Probability System (henceforth, LPS) over opponent's strategies with two features: (i) each strategy of the opponent is assigned positive probability by some component measure of the LPS; (ii) the first component measure of the LPS assigns positive probability only to strategies of the opponent which have survived the previous steps of the procedure. Brandenburger [3] proved that Permissibility coincides with the Dekel-Fudenberg procedure, i.e. the elimination of weakly dominated strategies, followed by the iterated elimination of strictly dominated strategies.

The aim of this paper is to provide an epistemic foundation of Permissibility. We base our analysis on two key notions: Cautious Rationality and Weak Belief. Cautious Rationality is the combination of lexicographic expected utility maximization and a cautious attitude of the player towards the primitive, payoff-relevant uncertainty. That is, each payoff-relevant event is assigned positive probability under some theory of the world that the player entertains, represented by some component measure of the player's LPS. Weak Belief of an event means that the player considers the event "infinitely more likely" than its complement. That is, the event is assigned probability one by the player's primary theory of the world, represented by the first component measure of the LPS. Weak Belief is based on the notion of "infinitely more likely than" between uncertain events due to Lo [11]. Roughly speaking, a player deems an event infinitely more likely than another one when she strictly prefers to bet on the first rather than on the second regardless of the size of the winning prizes for the two bets (given the same losing outcome). This notion of "infinitely more likely than" is weaker than the one due to Blume et al. [2]. In particular, the first is monotone, whereas the second one is not. Thus, the first one is suitable for the preference-based foundation of the monotone notion of Weak Belief, whereas the second one is not. Indeed, Brandenburger et al. [5], who adopt the notion of "infinitely more likely than" of [2] for their epistemic analysis of Iterated Admissibility, leave the epistemic foundations of Permissibility as an open question (see [5], page 333).

With this, we show that Permissibility characterizes the behavioral implications of Cautious Rationality and Common Weak Belief of Cautious Rationality in the canonical, universal type

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structure for LPS's. The canonical type structure represents all hierarchies of lexicographic beliefs on strategies, without imposing extraneous restrictions on players' beliefs.

The paper is structured as follows. Section 2 intoduces some preliminary technical concepts and the formal definitions of LPS's, type structures and type morphism. The main result in this section – the existence of a canonical, universal type structure for LPS's – is proved in our companion paper ([6]). Section 3 introduces the underlying game-theoretic framework, and the notions of Cautious Rationality and Weak Belief. In Section 4, we state and prove the main result. Appendix A illustrates the notion of "infinitely more likely than" and the preference-based foundation of Weak Belief. Appendix B provides the results about measurability of relevant sets.

2 Hierarchies of lexicographic beliefs and lexicographic type structures

2.1 Lexicographic probability systems

Given a topological space X, we denote by $\mathcal{M}(X)$ the set of Borel probability measures on X. We denote by $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) the set of all finite (resp. length-n) sequences of Borel probability measures on X, that is,

$$\begin{aligned}\mathcal{N}(X) &= \cup_{n\in\mathbb{N}}\mathcal{N}_n(X)\\ &= \cup_{n\in\mathbb{N}}(\mathcal{M}(X))^n.\end{aligned}$$

Each $\overline{\mu}=(\mu_1,...,\mu_n)\in\mathcal{N}(X)$ is called **lexicographic probability system** (LPS).

Suppose we are given topological spaces X and Y, and a Borel map $f:X\to Y$. The map $\widetilde{f}:\mathcal{M}(X)\to\mathcal{M}(Y)$, defined by

$$\widetilde{f}(\mu)(E)=\mu\left(f^{-1}(E)\right),\ \mu\in\mathcal{M}(X),\ E\in\Sigma_Y,$$

is called the image (or pushforward) measure map of f. For each $n\in\mathbb{N}$, the map $\widehat{f}_{(n)}:\mathcal{N}_n(X)\to\mathcal{N}_n(Y)$ is defined by

$$(\mu_1,...,\mu_n)\mapsto\widehat{f}_{(n)}((\mu_1,...,\mu_n))=\left(\widetilde{f}(\mu_k)\right)_{k\leq n}.$$

Thus the map $\widehat{f}:\mathcal{N}(X)\to\mathcal{N}(Y)$ defined by

$$\widehat{f}(\overline{\mu})=\widehat{f}_{(n)}(\overline{\mu}),\ \overline{\mu}\in\mathcal{N}_n(X),$$

is called the *image LPS map of* f. In other words, the map $\widehat{f}$ is the *combination* of the functions $\left(\widehat{f}_{(n)}\right)_{n\in\mathbb{N}}$, and it is Borel measurable.[1]

Furthermore, given topological spaces X and Y, we denote by Proj_X the canonical projection from $X\times Y$ onto X. Define the marginal measure of $\mu\in\mathcal{M}(X\times Y)$ on X as $\mathrm{marg}_X\mu=\widetilde{\mathrm{Proj}}_X(\mu)$. Consequently, the marginal of $\overline{\mu}\in\mathcal{N}(X\times Y)$ on X is defined by $\overline{\mathrm{marg}}_X\overline{\mu}=\widehat{\mathrm{Proj}}_X(\overline{\mu})$. Finally, we denote by Id_X the identity map on X, that is, $Id_X(x)=x$ for all $x\in X$.

[1]For details and proofs related to Borel measurability and continuity of the involved maps, the reader can consult [6].

2.2 Lexicographic type structures

The following definition is a natural generalization of the standard definition of epistemic type structures with beliefs represented by probability measures, i.e., length-1 LPS (cf. [10]). The formalism of lexicographic type strucures was first introduced by BFK ([5, Section 7]).[2]

Definition 1 *An* $(S_i)_{i\in I}$*-**based lexicographic type structure** is a structure* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$*, where*

1. *for each* $i \in I$*,* T_i *is a metrizable space;*
2. *for each* $i \in I$*, the function* $\beta_i : T_i \to \mathcal{N}\left(S_{-i} \times T_{-i}\right)$ *is measurable.*

We call each space T_i ***type space*** *and we call each* β_i ***belief map**.*[3] *Members of type spaces, viz.* $t_i \in T_i$*, are called **types**.*

Definition 2 *An* $(S_i)_{i\in I}$*-based lexicographic type structure* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *is*

- ***finite*** *if the cardinality of each type space* T_i *is finite;*
- ***compact*** *if each type space* T_i *is compact;*
- ***belief-complete*** *if each belief map* β_i *is onto;*
- ***continuous*** *if each belief map* β_i *is continuous.*

Analogous definitions hold if $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *is an LCPS type structure.*

The idea of belief-completeness was introduced by Brandenburger [4] and adapted to the present context. Note that each type space in a belief-complete type structure has the cardinality of the continuum. While finite type structures are trivially compact and continuous (but not belief-complete), the absence of an upper bound of the lenght of an LPS implies that a lexicographic type structure cannot be at the same time belief-complete, compact and continuous (for details, see [6]).

2.3 Type morphisms and universality

In what follows, given a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, we denote by T the Cartesian product of type spaces, that is, $T = \Pi_{i\in I} T_i$.

Definition 3 *Let* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *and* $\mathcal{T}' = \langle S_i, T'_i, \beta'_i \rangle_{i\in I}$ *be two* $(S_i)_{i\in I}$*-based lexicographic type structures. For each* $i \in I$*, let* $\varphi_i : T_i \to T'_i$ *be a measurable map such that*

$$\beta'_i \circ \varphi_i = \widehat{\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)} \circ \beta_i .$$

Then the function $(\varphi_i)_{i\in I} : T \to T'$ *is called **type morphism (from** $\mathcal{T}$ **to** $\mathcal{T}'$**).***

[2] Brandenburger et al. [5] defined lexicograpic types structures under the additional requirement that each belief is represented by LPS satisfying a, roughly speaking, disjoint supports condition, called *mutual singularity*.

[3] Observe that some authors ([1], [10]) use the terminology "type space" for what is called "type structure" here.

The notion of type morphism captures the idea that a type structure $\mathcal{T}$ is "contained in" another type structure $\mathcal{T}'$ if $\mathcal{T}$ can be mapped into $\mathcal{T}'$ in a way that preserves the beliefs associated with types. Condition (2) in the definition of type morphism expresses consistency between the function $\varphi_i : T_i \to T_i'$ and the induced function $\left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}\left(S_{-i} \times T_{-i}\right) \to \mathcal{N}\left(S_{-i} \times T_{-i}'\right)$. That is, the following diagram commutes:

$$\begin{array}{ccc} T_i & \xrightarrow{\beta_i} & \mathcal{N}(S_{-i} \times\ T_{-i}) \\ \Big\downarrow \varphi_i & & \Big\downarrow \left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right) \\ T_i' & \xrightarrow{\beta_i'} & \mathcal{N}(S_{-i} \times\ T_{-i}') \end{array} \tag{2.1}$$

The notion of type morphism does not make any reference to hierarchies of LPS's. But, as one should expect, the important property of type morphisms is that they preserve the explicit description of lexicographic belief hierarchies (for details, see [6])

We now ask: Is there a type structure into which any other type structure can be mapped? Alternatively put, since a type structure generates hierarchies of LPS's, does there exist a type structure that generates all hierarchies of beliefs? A type structure satisfying this requirement is called universal.

Definition 4 *An $(S_i)_{i\in I}$-based type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i\in I}$ is* ***universal*** *if for every other $(S_i)_{i\in I}$-based type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ there is a unique type morphism from $\mathcal{T}'$ to $\mathcal{T}$.*

In [6] we constructed the *canonical type structure for LPS's*, that is, a type structure in which types induce all possible hierarchies of lexicographic beliefs about the primitive uncertainty. Then, we showed that $\mathcal{T}_u$ is universal. This is in line with standard results on hierarchies of beliefs (cf., [12], [1]).

Theorem 1 (Catonini and De Vito, [6]) *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ be an arbitrary $(S_i)_{i\in I}$-based lexicographic type structure. Then, there exists a unique type morphism from $\mathcal{T}$ to $\mathcal{T}_u$. Thus $\mathcal{T}_u$ is a universal lexicographic type structure.*

The canonical type structure $\mathcal{T}_u$ is a particular instance of a belief-complete and continuous type structure. However, there exists other belief-complete type-structure, and they are not necessarily universal.

3 Permissibility, Cautiousness, and Weak Belief

3.1 Permissibility and the Dekel-Fudenberg Procedure

Consider a finite game $G = \langle I, (S_i, u_i)_{i\in I} \rangle$, where I is a two-player set and, for every $i \in I$, S_i is the set of strategies with $|S_i| \geq 2$ and $u_i : S \to \mathbb{R}$ is the payoff function. Define the expected payoff function π_i by extending u_i on $\mathcal{M}(S_i) \times \mathcal{M}(S_{-i})$ in the usual way:

$$\pi_i(\sigma_i, \sigma_{-i}) = \sum_{(s_i, s_{-i}) \in S_i \times S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}).$$

For any two vectors $x = (x_l)_{l=1}^n, y = (y_l)_{l=1}^n \in \mathbb{R}^n$, we write $x \geq_L y$ if either (1) $x_l = y_l$ for every $l \leq n$, or (2) there exists $m \leq n$ such that $x_m > y_m$ and $x_l = y_l$ for every $l < m$.

Definition 5 *A strategy $s_i \in S_i$ is* ***optimal under*** $\overline{\mu}_i = \left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}(S_{-i})$ *if*

$$\left(\pi_i(s_i, \mu_i^l)\right)_{l=1}^n \geq_L \left(\pi_i(s_i', \mu_i^l)\right)_{l=1}^n, \ \forall s_i' \in S_i.$$

We say s_i is a ***lexicographic best reply to*** $\overline{\mu}_i$ *on S_{-i} if it is optimal under $\overline{\mu}_i$. We denote by $r_i(\overline{\mu}_i)$ the set of player i's strategies which are optimal under $\overline{\mu}_i$.*

A strategy $s_i \in S_i$ is ***justifiable*** *if*

$$s_i \in \cup_{\overline{\mu}_i \in \mathcal{N}(S_{-i})} r_i(\overline{\mu}_i),$$

and ***cautiously justifiable*** *if*

$$s_i \in \cup_{\overline{\mu}_i \in \mathcal{N}^+(S_{-i})} r_i(\overline{\mu}_i).$$

Let $\mathcal{Q}$ denote the collection of all "Cartesian" subsets of S, i.e., subsets with the cross-product form $Q = Q_i \times Q_{-i}$, where $Q_i \subseteq S_i$ for every i.

Definition 6 *Fix a set $Q \in \mathcal{Q}$. A strategy $s_i \in S_i$ is* ***admissible with respect to*** *Q if and only if there exists $\sigma_{-i} \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\sigma_{-i} = Q_{-i}$ and $\pi_i(s_i, \sigma_{-i}) \geq \pi_i(s_i', \sigma_{-i})$ for every $s_i' \in Q_i$. If strategy $s_i \in S_i$ is admissible with respect to $S_i \times S_{-i}$, we simply say that s_i is admissible.*

Remark 1 *Fix $Q \in \mathcal{Q}$. A strategy $s_i \in S_i$ is* ***weakly dominated with respect to*** *Q if there exists $\sigma_i \in \mathcal{M}(S_i)$ with $\sigma_i(Q_i) = 1$ such that $\pi_i(\sigma_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for every $s_{-i} \in Q_{-i}$ and $\pi_i(\sigma_i, s_{-i}') > \pi_i(s_i, s_{-i}')$ for some $s_{-i}' \in Q_{-i}$. A standard result ([**?**, Lemma 4]) states that a strategy $s_i \in S_i$ is not weakly dominated with respect to Q if and only if it is admissible with respect to Q.*

Definition 7 *Fix a set $Q \in \mathcal{Q}$. A strategy $s_i \in S_i$ is* ***strictly dominated with respect to*** *Q if there exists $\sigma_i \in \mathcal{M}(S_i)$ such that $\pi_i(\sigma_i, s_{-i}') > \pi_i(s_i, s_{-i}')$ for some $s_{-i}' \in Q_{-i}$. If strategy $s_i \in S_i$ is strictly dominated with respect to $S_i \times S_{-i}$, we simply say that s_i is strictly dominated.*

Remark 2 *Fix a set $Q \in \mathcal{Q}$. By [**?**, Lemma 2], a strategy $s_i \in S_i$ is* ***strictly dominated with respect to*** *Q if and only if there exists $\sigma_{-i} \in \mathcal{M}(S_{-i})$ such that $\sigma_{-i}(Q_{-i}) = 1$ and $\pi_i(s_i, \sigma_{-i}) \geq \pi_i(s_i', \sigma_{-i})$ for every $s_i' \in Q_i$.*

Fix $Q \in \mathcal{Q}$. We write $\mathrm{WD}_i(Q)$ (resp., $\mathrm{ND}_i(Q)$) for the set of not weakly (resp., strictly) dominated strategies of player i with respect to Q. Let $\mathrm{WD}(Q) = \prod_{i \in I} \mathrm{WD}_i(Q)$, $\mathrm{ND}(Q) = \prod_{i \in I} \mathrm{ND}_i(Q)$, $\mathrm{ND}^1(Q) = \mathrm{ND}(Q)$. For each $k > 1$, let $\mathrm{ND}^k(Q) = \mathrm{ND}(\mathrm{ND}^{k-1}(Q))$.

The sets

$$\mathrm{WD}(S);$$
$$\mathrm{ND}^k(\mathrm{WD}(S)), \ \forall k = 1, 2, ...$$

constitute the *Dekel-Fudenberg procedure* ([8])

We define the following sets:

$$\rho_i^W(Q_{-i}) = \left\{s_i \in S_i : \exists \overline{\mu}_i \in \mathcal{N}^+(S_{-i}), \mu_i^1(Q_{-i}) = 1 \wedge s_i \in r_i(\overline{\mu}_i)\right\},$$

$$\rho^W(Q) = \prod_{i\in I} \rho_i^W(Q_{-i}).$$

The interpretation is as follows: $\rho^W(Q)$ *is the set of cautiously justifiable strategy profiles that could be chosen by the players if every* i *weakly believes that the co-player chooses in* Q_{-i}. Therefore, we call the mapping $\rho^W : \mathcal{Q} \to \mathcal{Q}$ "**weak rationalization operator**."[4] Note that $\rho^W(\emptyset) = \emptyset$.

Remark 3 *The operator* ρ^W *is* monotone*: for every pair of subsets* $E, F \in \mathcal{Q}$*, if* $E \subseteq F$ *then* $\rho^W(E) \subseteq \rho^W(F)$*.*

We define the k-th iteration of ρ^W (the k-fold composition of ρ with itself) recursively as follows. For each $Q \in \mathcal{Q}$, define $\rho^{W,0}(Q) = Q$ for convenience; then for each $k \geq 1$,

$$\rho^{W,k}(Q) = \rho^W(\rho^{W,k-1}(Q)).$$

Note that, by the monotonicity of ρ^W, the sequence of subsets $(\rho^{W,k}(S))_{k=1}^{\infty}$ is weakly decreasing, i.e., $\rho^{W,k+1}(S) \subseteq \rho^{W,k}(S)$ $(k \in \mathbb{N})$. Therefore define:

$$\rho^{W,\infty}(S) = \bigcap_{k\geq 1} \rho^{W,k}(S).$$

Since each strategy set S_i is finite, there exists $M \in \mathbb{N}$ such that $\rho^{W,\infty}(S) = \rho^{W,M}(S) \neq \emptyset$. For notational convenience, for each $m = 0, ..., \infty$, let $S^m = \rho^{W,m}(S)$ and, for each $i \in I$, $S_i^m = \mathrm{Proj}_{S_i}\rho^{W,m}(S)$.

Definition 8 (Brandenburger, [3]) *A strategy profile* $s \in S$ *is* ***permissible*** *if* $s \in S^\infty = \rho^{W,\infty}(S)$*.*

Theorem 2 (Brandenburger, [3]) *Fix a finite* $G = \langle I, (S_i, u_i)_{i\in I}\rangle$*. Thus*

$$\begin{aligned} \rho^{W,1}(S) &= S^1 = \mathrm{WD}(S), \\ \rho^{W,k}(S) &= S^k = \mathrm{ND}^{k-1}(\mathrm{WD}(S)), \forall k = 2, 3, ... \end{aligned}$$

Therefore, a strategy profile is permissible if and only if it survives the Dekel-Fudenberg procedure.

[4] The weak rationalization operator represents an example of **justification operator**, a concept which was first explicitly presented by Milgrom and Roberts [**?**].

3.2 Rationality, Cautiousness and Weak Belief

Append to the game G a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$.

Definition 9 *A type $t_i \in T_i$ is **cautious (in $\mathcal{T}$)** if $\overline{\text{marg}}_{S_{-i}} \beta_i(t_i) \in \mathcal{N}^+ (S_{-i})$.*

Thus, for strategy-type pairs we define the following notions.

Definition 10 *Fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$.*

1. *Say (s_i, t_i) is **rational (in $\mathcal{T}$)** if s_i is optimal under $\overline{\text{marg}}_{S_{-i}} \beta_i(t_i)$. Let R_i be the set of all rational $(s_i, t_i) \in S_i \times T_i$.*
2. *Say (s_i, t_i) is **cautiously rational (in $\mathcal{T}$)** if it is rational and $(s_i, t_i) \in C_i$.*

Cautious rationality has a convenient invariance property under type morphisms between type structures. The following results state this formally.

Lemma 1 (Catonini and De Vito, [7]) *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ be lexicographic type structures, so that there exists a type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Fix a type $t_i \in T_i$. Thus*

(i) *t_i is cautious if and only if $\varphi_i(t_i)$ is cautious.*

(ii) *A strategy-type pair (s_i, t_i) is rational in $\mathcal{T}$ if and only $(s_i, \varphi_i(t_i))$ is rational in $\mathcal{T}^*$.*

Next, the notion of LPS-based notion of Weak Belief.

Definition 11 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and a non-empty event $E \subseteq S_{-i} \times T_{-i}$. Fix also $t_i \in T_i$ with $\beta_i(t_i) = (\mu^1, ..., \mu^n)$. We say that E is **weakly believed under** $\beta_i(t_i)$ if $\mu^1(E) = 1$. We say that $t_i \in T_i$ **weakly believes** E if E is weakly believed under $\beta_i(t_i)$.*

The notion of Weak Belief captures the idea that event E is "infinitely more likely than" its complement: see Appendix A for its preference-based treatment. Weak Belief satisfies the following properties.

Property 1: (Marginalization) If E is a non-empty event in $S_{-i} \times T_{-i}$ which is weakly believed under $\overline{\mu}_i$, then $\text{Proj}_{S_{-i}}(E)$ is weakly believed under $\overline{\text{marg}}_{S_{-i}} \overline{\mu}_i$.

Property 2: (Conjunction and Disjunction) Fix non-empty events $E_1, E_2, ...$ in $S_{-i} \times T_{-i}$. Suppose that, for each k, E_k is weakly believed under $\overline{\mu}_i$. Thus $\cap_k E_k$ and $\cup_k E_k$ are weakly believed under $\overline{\mu}_i$.

Property 3: (Monotonicity) Fix a non-empty event $E \subseteq S_{-i} \times T_{-i}$ which is weakly believed under $\overline{\mu}_i$. If event $F \subseteq S_{-i} \times T_{-i}$ is such that $E \subseteq F$, then F is weakly believed under $\overline{\mu}_i$.

For each player $i \in I$, let $\mathbf{WB}_i : \Sigma_{S_{-i}\times T_{-i}} \to \Sigma_{S_i \times T_i}$ be the operator defined by

$$\mathbf{WB}_i\left(E_{-i}\right) = \left\{(s_i, t_i) \in S_i \times T_i \left| t_i \text{ weakly believes } E_{-i}\right.\right\}, \; E_{-i} \in \Sigma_{S_{-i}\times T_{-i}}.$$

Corollary D.1 in Appendix B shows that the set $\mathbf{WB}_i\left(E_{-i}\right)$ is Borel in $S_i \times T_i$ for every event $E_{-i} \subseteq S_{-i} \times T_{-i}$; so the operator $\mathbf{WB}_i : \Sigma_{S_{-i}\times T_{-i}} \to \Sigma_{S_i\times T_i}$ is well-defined.

The Weak Belief operator $\mathbf{WB}_i$ has invariance properties under type morphisms between type structures which are analogous to the ones of (cautious) rationality

Lemma 2 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^*\rangle_{i\in I}$ be lexicographic type structures such that there exists a type morphism $(\varphi_i)_{i\in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Let $E_{-i} \subseteq S_{-i} \times T_{-i}$ and $E^*_{-i} \subseteq S_{-i} \times T^*_{-i}$ be non-empty events such that $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)\left(E_{-i}\right) \subseteq E^*_{-i}$. Then $\left(\mathrm{Id}_{S_i}, \varphi_i\right)\left(\mathbf{WB}_i\left(E_{-i}\right)\right) \subseteq \mathbf{WB}_i\left(E^*_{-i}\right)$.*

Proof. Fix $(s_i, t_i) \in \mathbf{WB}_i\left(E_{-i}\right)$. Write $\beta_i(t_i) = (\mu_1, ..., \mu_n)$ and $(\beta_i^* \circ \varphi_i)(t_i) = (\widehat{\mu}_1, ..., \widehat{\mu}_n)$. Since $\beta_i^* \circ \varphi_i = \widehat{\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)} \circ \beta_i$, it is easy to observe that $\mu_1(E_{-i}) = 1$ implies $\widehat{\mu}_i(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)\left(E_{-i}\right)) = 1$. That is, $\beta_i^*(\varphi_i(t_i))$ weakly believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)\left(E_{-i}\right)$. Thus, by monotonicity of weak belief, $\beta_i^*(\varphi_i(t_i))$ weakly believes also $E^*_{-i} \supseteq \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)\left(E_{-i}\right)$. ■

4 Common Weak Belief of Cautious Rationality and the main result

We now provide an epistemic foundation of permissibility in "sufficiently rich" (i.e., belief-complete) type structures. In what follows, fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ and, for each player $i \in I$, let R_i^1 be the set of all cautiously rational strategy-type pairs $(s_i, t_i) \in S_i \times T_i$. For each $m > 1$, define R_i^m inductively by

$$R_i^{m+1} = R_i^m \cap \mathbf{WB}_i\left(R_{-i}^m\right).$$

We write $R_i^0 = S_i \times T_i$ and $R_i^\infty = \cap_{m\in\mathbb{N}} R_i^m$ for each $i \in I$. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^{m+1}$, we say that there is **cautious rationality and mth-order weak belief of cautious rationality** ($\mathbf{R}^{\mathrm{c}}m\mathbf{WBR}^{\mathrm{c}}$) at this state. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^\infty$, we say that there is **cautious rationality and common weak belief of cautious rationality** ($\mathbf{R}^{\mathrm{c}}\mathbf{CWBR}^{\mathrm{c}}$) at this state.

Note that, for each $m > 1$,

$$R_i^{m+1} = R_i^1 \cap \left(\cap_{l\leq m} \mathbf{WB}_i\left(R_{-i}^l\right)\right),$$

and each R_i^m is Borel in $S_i \times T_i$ (see Appendix B).

We now state the main results of this paper.

Theorem 3 *Fix a game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$ and an associated belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$. The following statements hold:*

(i) *for each $m \geq 0$, $\prod_{i\in I} \mathrm{Proj}_{S_i}\left(R_i^m\right) = \prod_{i\in I} S_i^m$;*

(ii) *if $\mathcal{T}$ is universal, then $\prod_{i\in I} R_i^\infty \neq \emptyset$ and $\prod_{i\in I} \mathrm{Proj}_{S_i}\left(R_i^\infty\right) = \prod_{i\in I} S_i^\infty$.*

The proof of Theorem 3 will make use of the following results.

Lemma 3 *Fix a game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$ and an associated type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$. The following statements hold:*
(i) for each $m \geq 1$, $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^m) \subseteq \prod_{i\in I} S_i^m$;
(ii) $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^\infty) \subseteq \prod_{i\in I} S_i^\infty$.

Proof: The proof of part (i) is by induction on m.

($m = 1$) Fix $i \in I$. Let $s_i \in \mathrm{Proj}_{S_i}\left(R_i^1\right)$, so that $(s_i, t_i) \in R_i^1$ for some $t_i \in T_i$. Then s_i is optimal against $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$, that is, $s_i \in S_i^1$. So $\mathrm{Proj}_{S_i}\left(R_i^1\right) \subseteq S_i^1$ for each $i \in I$.

($m \geq 2$) Suppose that the statement has been shown to hold for all $l = 1, ..., m-1$. We show that the statement is true for $l = m$.

Fix a player $i \in I$, and let $s_i \in \mathrm{Proj}_{S_i}(R_i^m)$, so that $(s_i, t_i) \in R_i^m$ for some $t_i \in T_i$. It follows from the definition of R_i^m that $(s_i, t_i) \in R_i^{m-1}$, so, by the induction hypothesis, $s_i \in S_i^m$. Also, R_{-i}^{m-1} is weakly believed under $\beta_i(t_i) = \left(\mu_i^1, ..., \mu_i^n\right)$, hence

$$\mathrm{marg}_{S_{-i}}\mu_i^1\left(S_{-i}^{m-1}\right) \geq \mathrm{marg}_{S_{-i}}\mu_i^1\left(\mathrm{Proj}_{S_{-i}}\left(R_{-i}^{m-1}\right)\right) \geq \mu_i^1\left(R_{-i}^{m-1}\right) = 1,$$

where the first inequality follows from the induction hypothesis. Hence s_i is optimal against $\mathrm{marg}_{S_{-i}}\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}^+(S_{-i})$ with $\mu_i^1(S_{-i}^{m-1}) = 1$, that is, $s_i \in S_i^m$. So $\mathrm{Proj}_{S_i}(R_i^m) \subseteq S_i^m$ for each $i \in I$.

This concludes the proof of part (i). Part (ii) follows from part (i). ■

Lemma 4 *Fix a game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$. There exists a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^*\rangle_{i\in I}$ such that:*
(i) for each $m \geq 1$, $\prod_{i\in I} \mathrm{Proj}_{S_i}\left(R_i^{*,m}\right) = \prod_{i\in I} S_i^m$;
(ii) $\prod_{i\in I} \mathrm{Proj}_{S_i}\left(R_i^{*,\infty}\right) = \prod_{i\in I} S_i^\infty$.

Proof: Let M be the smallest natural number such that $\prod_{i\in I} S_i^\infty = \prod_{i\in I} S_i^M$. By definition of Permissibility, for each $i \in I$, $k = 1, ..., M-1$, and $s_i \in S_i^k \backslash S_i^{k+1}$, we can pick $\overline{\mu}_{s_i} = (\mu_{s_i}^1, ..., \mu_{s_i}^n) \in \mathcal{N}^+(S_{-i})$ such that $\mu^1(S_{-i}^{k-1}) = 1$ and s_i is optimal under μ_{s_i}. For each $s_i \in S_i^M$, we can pick $\overline{\mu}_{s_i} = (\mu_{s_i}^1, ..., \mu_{s_i}^n) \in \mathcal{N}^+(S_{-i})$ such that $\mu_{s_i}^1(S_{-i}^M) = 1$ and s_i is optimal under μ_{s_i}.

Now we construct a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^*\rangle_{i\in I}$.

For each $i \in I$, let T_i^* be a copy of S_i. For each $s_i \in S_i$, we will denote the corresponding type as t_{s_i}. Let $\psi_i : S_i \to (S_i \times T_i^*)$ be a map that associates each strategy $s_i \in S_i$ with the strategy-type pair (s_i, t_{s_i}).

For each $i \in I$, we define the belief map $\beta_i^* : T_i^* \to \mathcal{N}(S_{-i} \times T_{-i}^*)$ as follows. For each $s_i \in S_i^0 \backslash S_i^1$, let $\beta_i^*(t_{s_i})$ be arbitrary. For each $s_i \in S_i^1$, let $\beta_i^*(t_{s_i}) = \widehat{\psi}_{-i}(\overline{\mu}_{s_i})$.

Note that for every $i \in I$ and $s_i \in S_i^1$, t_{s_i} is cautious.

Lemma 3.(i) entails that, for each $i \in I$ and $m \geq 1$, if $(s_i, t_i) \in R_i^{*,m}$ then $s_i \in S_i^m$. Conversely, if $s_i \in S_i^1$, $(s_i, t_{s_i}) \in R_i^{*,1}$ because t_{s_i} is cautious and s_i is optimal under $\overline{\mathrm{marg}}_{S_{-i}}\beta_i^*(t_{s_i}) = \overline{\mu}_{s_i}$. Fix $m > 1$ and assume by induction that for every $j \in I$ and $s_j \in S_j^{m-1}$, $\left(s_j, t_{s_j}\right) \in R_j^{*,m-1}$. Fix $s_i \in S_i^m$. Write $\beta_i^*(t_{s_i}) = (\mu_i^1, ..., \mu_i^n)$. By the inductive hypothesis, $(s_i, t_{s_i}) \in R_i^{*,m-1}$, and for every $s_{-i} \in S_{-i}^{m-1}$, $\left(s_{-i}, t_{s_{-i}}\right) \in R_{-i}^{*,m-1}$. Then we have

$$\mu_i^1(R_{-i}^{*,m-1}) \geq \mu_i^1\left(\left\{\left(s_{-i}, t_{s_{-i}}\right) \in S_{-i} \times T_{-i}^* \,\middle|\, s_{-i} \in S_{-i}^{m-1}\right\}\right) = \mu_{s_i}^1(S_{-i}^{m-1}) = 1,$$

where for $m = M$ the last equality comes from $\mu^1_{s_i}(S^M_{-i}) = 1$ and $S^{M-1}_{-i} \supset S^M_{-i}$. Thus, t_{s_i} weakly believes $R^{*,m-1}_{-i}$. This establishes part (i).

Fix $s_i \in S^\infty_i$. We have just shown that $(s_i, t_{s_i}) \in R^{*,m}_i$ for all $m \geq 1$. Then, $(s_i, t_{s_i}) \in R^{*,\infty}_i$. This establishes part (ii). ■

Proof of Theorem 3: Part (i): The statement is trivially true for $m = 0$.

Fix $m \geq 1$. Suppose that the statement has been shown to hold for all $l = 1, ..., m-1$. We show that the statement is true for $l = m$.

Fix a player $i \in I$. Lemma 3.(i) gives that $\mathrm{Proj}_{S_i}(R^m_i) \subseteq S^m_i$. Conversely, let $s_i \in S^m_i$. So there is $\overline{\nu}_i = (\nu^1_i, ..., \nu^n_i) \in \mathcal{N}^+(S_{-i})$ such that $\nu^1_i(S^{m-1}_{-i}) = 1$, and s_i is a lexicographic best reply to $\overline{\nu}_i$. We now show the existence of an LPS $\overline{\mu}_i = (\overline{\mu}^1_i, ..., \overline{\mu}^n_i) \in \mathcal{N}(S_{-i} \times T_{-i})$ such that

(a) $\overline{\mathrm{marg}}_{S_{-i}}\overline{\mu}_i = \overline{\nu}_i$; and

(b) $(R^k_{-i})^{m-1}_{k=0}$ are weakly believed under $\overline{\mu}_i$.

To this end, note that, by the induction hypothesis, for each $s_{-i} \in S^{m-1}_{-i}$ there exists $t_{s_{-i}} \in T_{-i}$ such that $(s_{-i}, t_{s_{-i}}) \in R^{m-1}_{-i}$. Fix some $t^0_{-i} \in T_{-i}$, and define the map $\psi^{m-1}_{-i} : S_{-i} \to S_{-i} \times T_{-i}$ as

$$\psi^{m-1}_{-i}(s_{-i}) = \begin{cases} (s_{-i}, t_{s_{-i}}), & \text{if } s_{-i} \in S^{m-1}_{-i}, \\ (s_{-i}, t^0_{-i}), & \text{if } s_i \in S_{-i} \setminus S^{m-1}_{-i}, \end{cases}$$

(Of course, the map ψ^{m-1}_{-i} is continuous, since strategy sets are endowed with the discrete topology.) Define $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times T_{-i})$ by $\overline{\mu}_i = \widehat{\psi}^{m-1}_{-i}(\overline{\nu}_i)$. It readily follows that $\overline{\mu}_i$ satisfies property (a), since $\mathrm{Proj}_{S_{-i}} \circ \psi^{m-1}_{-i} = \mathrm{Id}_{S_{-i}}$. Property (b) also holds, in that

$$\mu^1_i(R^{m-1}_{-i}) = \nu^1_i((\psi^{m-1}_{-i})^{-1}(R^{m-1}_{-i})) = \nu^1_i(S^{m-1}_{-i}) = 1, \tag{4.1}$$

where the second equality comes from the the induction hypothesis. So, R^{m-1}_{-i} is weakly believed under $\overline{\mu}_i$. By monotonicty of weak beliefs, then for each $k < m-1$, R^k_{-i} is weakly believed under $\overline{\mu}_i$ too. It now follows from belief-completeness that there is $t_i \in T_i$ such that $\beta_i(t_i) = \overline{\mu}_i$; this implies $(s_i, t_i) \in R^m_i$, hence $s_i \in \mathrm{Proj}_{S_i}(R^m_i)$.

Part (ii): Fix a player $i \in I$. Lemma 3.(ii) gives that $\mathrm{Proj}_{S_i}(R^\infty_i) \subseteq S^\infty_i$. Conversely, suppose that $\mathcal{T}$ is universal (for instance, $\mathcal{T}_u$). Then, by Lemma 4, there exists a finite type structure $\mathcal{T}^* = \langle S_i, T^*_i, \beta^*_i \rangle_{i \in I}$ such that, for each $i \in I$ and each $m \geq 1$,

(a) $\mathrm{Proj}_{S_i}(R^{*,m}_i) = S^m_i$,

(b) $\mathrm{Proj}_{S_i}(R^{*,\infty}_i) = S^\infty_i$.

Then, for every $s_i \in S^\infty_i$, there exists $t_i \in T^*_i$ such that $(s_i, t_i) \in R^{*,m}_i$ for all $m \in \mathbb{N}$. It thus follows from Lemma 2 that $(\mathrm{Id}_{S_i}, d_i)((s_i, t_i)) \in R^m_i$ for all $m \in \mathbb{N}$. Hence $(\mathrm{Id}_{S_i}, d_i)((s_i, t_i)) \in R^\infty_i$. Consequently $S^\infty_i \subseteq \mathrm{Proj}_{S_i}(R^\infty_i) \neq \emptyset$. The conclusion follows. ■

Appendix A: Preference-based representation of Weak-Belief

Fix a lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, where each strategy set S_i is finite. To shorten notation, it will be convenient to set $\Omega = S_{-i} \times T_{-i}$ and to drop i's subscript from LPS's $\overline{\mu}_i$ on Ω.

An act on Ω is a Borel measurable function $f : \Omega \to [0, 1]$. We denote by $\mathrm{ACT}(\Omega)$ the set of all acts on Ω. A decision maker has preferences over elements of $\mathrm{ACT}(\Omega)$. For $x \in [0, 1]$, write $\overrightarrow{x}$ for the constant act associated with x, i.e., $\overrightarrow{x}(\omega) = x$ for all $\omega \in \Omega$. Each constant act is identified with the associated outcome in a natural way. In what follows, we assume that the outcome space $[0,1]$ is in utils. Given a Borel set $E \subseteq \Omega$ and acts $f, g \in$ACT(Ω), let $(f_E, z_{\Omega\backslash E}) \in$ACT$(\Omega)$ be defined as follows:

$$(f_E, z_{\Omega\backslash E})(\omega) = \begin{cases} f(\omega), & \omega \in E, \\ z(\omega), & \omega \in \Omega\diagdown E. \end{cases}$$

Let $\succsim$ be a preference relation on ACT(Ω) and write $\succ$ (resp. $\sim$) for strict preference (resp. indifference). The preference relation $\succsim$ sastisfies the following axioms:

Axiom 1 Order: $\succsim$ is a complete, transitive, reflexive binary relation on ACT(Ω).

Axiom 2 Independence: For all $f, g, z \in$ACT(Ω) and $\alpha \in (0,1]$,

$$\begin{aligned} f &\succ g \text{ implies } \alpha f + (1-\alpha)\, z \succ \alpha g + (1-\alpha)\, z, \text{ and} \\ f &\sim g \text{ implies } \alpha f + (1-\alpha)\, z \sim \alpha g + (1-\alpha)\, z. \end{aligned}$$

Moreover, let $\succsim_E$ denote the *conditional preference* given E, that is, $f \succsim_E g$ if and only if $(f_E, z_{\Omega\backslash E}) \succsim (g_E, z_{\Omega\backslash E})$ for some $z \in$ACT(Ω). Standard results (see [2, p. 64] for a proof) state that, under Axioms 1 and 2, $(f_E, z_{\Omega\backslash E}) \succsim (g_E, z_{\Omega\backslash E})$ holds for all $z \in$ACT(Ω) if it holds for some z.

Throughout, we mantain the assumption that $\overline{\mu}$ is a LEU representation of $\succsim$, i.e., $\succsim = \succsim^{\overline{\mu}}$. (This makes sense, since each LEU representation satisfies Axioms 1 and 2.)

Recall that an event $E \subseteq \Omega$ is **Savage-null** under $\succsim$ if $f \sim_E g$ for all $f, g \in$ACT(Ω). Say that E is **non-null** under $\succsim$ if it is not Savage-null under $\succsim$. Say that event $E \subseteq \Omega$ is **fully-believed** under $\succsim$ if $f \sim_{\Omega\backslash E} g$ for all $f, g \in$ACT(Ω). We thus have:

Proposition A.1 *Fix an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. An event $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$ if and only if $\mu^l(E) = 0$ for all $l \leq n$.*

Proof: If $\mu^l(E) = 0$ for all $l \leq n$, then obviously $f \sim_E^{\overline{\mu}} g$ for all $f, g \in$ACT(Ω). On the other hand, if $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$, then $\overrightarrow{1} \sim_E^{\overline{\mu}} \overrightarrow{0}$. That is,

$$\left(\int_E d\mu^l + \int_{\Omega\backslash E} z d\mu^l\right)_{l=1}^n = \left(0 + \int_{\Omega\backslash E} z d\mu^l\right)_{l=1}^n, \forall z \in \text{ACT}(\Omega),$$

which implies $\mu^l(E) = 0$ for all $l \leq n$. ■

Corollary A.1 *Fix an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is fully-believed under $\succsim^{\overline{\mu}}$ if and only if $\mu^l(E) = 1$ for all $l \leq n$.*

Definition A.1 *Fix events $E, F \subseteq \Omega$ with $E \neq \emptyset$. Say that E is* ***more likely*** *than F if for all $x, y \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\diagdown E}) \succsim^{\overline{\mu}} (\overrightarrow{x}_F, \overrightarrow{y}_{\Omega\diagdown F}).$$

Say that E is deemed ***infinitely more likely*** *than F (Lo, [11]), and write $E \gg F$, if for all $x, y, z \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\diagdown E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\diagdown F}).$$

In words, E is more likely than F if the Decision Maker prefers to bet on E rather than on F given the same prizes for the two bets; this choice theoretic notion is due to Savage [13]. On the other hand, E is infinitely more likely than F if the Decision Maker *strictly* prefers to bet on E rather than on F, and increasing the prize by any extent for the second bet does not induce his strict preference to change.

Note that, according to Definition A.1, event F may, but *need not*, be Savage-null under $\succsim^{\overline{\mu}}$ if $E \gg F$. When $\succsim^{\overline{\mu}}$ is represented by the Subjective Expected Utility model, $E \gg F$ implies that F is Savage-null. It is also noteworthy that this definition of "infinitely more likely than" involves only binary acts, and the events in the definition need not be pairwise disjoint.

Given an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty event $E \subseteq \Omega$, let

$$\mathcal{I}_E = \inf \left\{ l \in \{1, ...n\} \middle| \mu^l(E) > 0 \right\}.$$

Proposition A.2 *Fix disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$.*

1. *E is more likely than F if and only if*

$$\left(\mu^l(E)\right)_{l=1}^n \geq_L \left(\mu^l(F)\right)_{l=1}^n.$$

2. *$E \gg F$ if and only if $\mathcal{I}_E < \mathcal{I}_F$.*

Proof: Part 1: Let $x, y \in [0,1]$ with $x > y$. The statement follows from the following chain of logically equivalent relations.

$$\begin{aligned}
(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\setminus E}) \quad \succsim \quad & ^{\overline{\mu}}(\overrightarrow{x}_F, \overrightarrow{y}_{\Omega\setminus F}) \Longleftrightarrow \left(\int_E x d\mu^l + \int_{\Omega\setminus E} y d\mu^l\right)_{l=1}^n \geq_L \left(\int_F x d\mu^l + \int_{\Omega\setminus F} y d\mu^l\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(x\mu^l(E) + y\mu^l(\Omega\setminus E)\right)_{l=1}^n \geq_L \left(x\mu^l(F) + y\mu^l(\Omega\setminus F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(x\mu^l(E) + y - y\mu^l(E)\right)_{l=1}^n \geq_L \left(x\mu^l(F) + y - y\mu^l(F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left((x-y)\mu^l(E)\right)_{l=1}^n \geq_L \left((x-y)\mu^l(F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(\mu^l(E)\right)_{l=1}^n \geq_L \left(\mu^l(F)\right)_{l=1}^n.
\end{aligned}$$

Part 2: See Proposition A.4 in [7] ■

The likelihood relation $\gg$ possesses a "monotonicity" property, as the following result shows.

Proposition A.3 (Catonini and De Vito, [7]) *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$. Let $E_1 \subseteq \Omega$ be a non-empty event such that $E_1 \subseteq E$. Thus, if $E_1 \gg F$ then $E \gg F$.*

Corollary A.2 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and pairwise disjoint, non-empty events $E, F, G \subseteq \Omega$. If $E \gg G$, then $E \cup F \gg G$.*

The notion of **Weak Belief** for an event $E \subseteq \Omega$ simply requires that E be "infinitely more likely than" not-E.

Definition A.2 *Fix a non-empty event $E \subseteq \Omega$. Say that E is **weakly believed under** $\succsim^{\overline{\mu}}$ if $E \gg \Omega\setminus E$.*

Theorem A.1 *Fix an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and a non-empty event $E \subseteq \Omega$. Thus E is weakly believed under $\succsim^{\overline{\mu}}$ if and only if $\mu^1(E) = 1$.*

Proof: Fix $x, y, z \in [0,1]$ with $x > y$. We have:

$$\begin{aligned}
(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\setminus E}) \succ^{\overline{\mu}} (\overrightarrow{z}_{\Omega\setminus E}, \overrightarrow{y}_E) &\iff \left(\int_E x d\mu^l + \int_{\Omega\setminus E} y d\mu^l\right)_{l=1}^n >_L \left(\int_{\Omega\setminus E} z d\mu^l + \int_E y d\mu^l\right)_{l=1}^n \\
&\iff \left(x\mu^l(E) + y\mu^l(\Omega\setminus E)\right)_{l=1}^n >_L \left(z\mu^l(\Omega\setminus E) + y\mu^l(E)\right)_{l=1}^n \\
&\iff \left(x\mu^l(E) + y - y\mu^l(E)\right)_{l=1}^n >_L \left(z\mu^l(\Omega\setminus E) + y - y\mu^l(\Omega\setminus E)\right)_{l=1}^n \\
&\iff \left((x-y)\mu^l(E)\right)_{l=1}^n >_L \left((z-y)\mu^l(\Omega\setminus E)\right)_{l=1}^n .
\end{aligned}$$

Since $x > y$, $\mu^1(E) = 1$ implies

$$(x-y)\mu^1(E) > (z-y)\mu^1(\Omega\setminus E) = 0.$$

Suppose now that $\mu^1(E) \neq 1$. Then, for $y = 0$, $x = \frac{1}{2}\mu^1(\Omega\setminus E) > 0$ and $z = \mu^1(E)$, we have:

$$(x-y)\mu^1(E) = \frac{1}{2}\mu^1(\Omega\setminus E)\mu^1(E) < \mu^1(E)\mu^1(\Omega\setminus E) = (z-y)\mu^1(\Omega\setminus E).$$

■

5 Appendix B: Proof of measurability of the relevant sets

The aim of this Section is to show that, for a given type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, the sets R_i^m, $m > 1$, as defined in the main text, are Borel subsets of $S_i \times T_i$. We do this by first showing that $\mathbf{WB}_i(E) \subseteq S_i \times T_i$ is Borel for every event $E \subseteq S_{-i} \times T_{-i}$.

Lemma D.1 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and non-empty event $E \subseteq S_{-i} \times T_{-i}$. Thus, the set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ under which E is weakly believed is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$.*

Proof: By [**?**, Theorem 17.24] it follows that, for a given event $E \subseteq S_{-i} \times T_{-i}$, the set of probability measures μ satisfying $\mu(E) = p$ for $p \in \mathbb{Q} \cap [0,1]$ is measurable in $\mathcal{M}(S_{-i} \times T_{-i})$. So the sets of all $\mu \in \mathcal{M}(S_{-i} \times T_{-i})$ satisfying $\mu(E) = 1$ are Borel in $\mathcal{M}(S_{-i} \times T_{-i})$. Now, fix n. By the above argument and by definition of $\mathcal{N}_n(S_{-i} \times T_{-i})$, it turns out that the set

$$\begin{aligned}
\mathcal{W}_n^1 &= \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) \,\middle|\, \mu^1(E) = 1\right\} \\
&= \left\{\mu \in \mathcal{M}(S_{-i} \times T_{-i}) \,\middle|\, \mu^l(E) = 1\right\} \times (\mathcal{M}(S_{-i} \times T_{-i}))^{n-1}
\end{aligned}$$

is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. The set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ under which E is weakly believed is given by $\cup_{n\in\mathbb{N}} \mathcal{W}_n^1$, so Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. ■

By the measurability of each belief map in a lexicographic type structures, it follows that

Corollary D.1 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$. Thus, for every $i \in I$, if $E \subseteq S_{-i} \times T_{-i}$ is a non-empty event, then $\mathbf{WB}_i(E)$ is a Borel subset of $S_i \times T_i$.*

The measurability of Cautious Rationality has already been established in [7].

Lemma D.2 (Catonini and De Vito, [7]) *Fix a type structure* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$*. Thus, for every* $i \in I$*,* R_i^1 *is Borel in* $S_i \times T_i$*.*

We can now state and prove the desired result:

Lemma D.5 *Fix a type structure* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$*. Thus, for each* $i \in I$ *and* $m \geq 1$*,* R_i^m *is Borel in* $S_i \times T_i$*.*

Proof: By Lemma D.1**,** for each $i \in I$, the set R_i^1 is Borel in $S_i \times T_i$. Note that $R_i^{m+1} = R_i^m \cap \mathbf{WB}_i \left(R_{-i}^m \right)$. By Corollary D.1, the set $\mathbf{WB}_i \left(R_{-i}^m \right)$ is Borel in $S_i \times T_i$ provided that R_{-i}^m is Borel. The conclusion follows from an easy induction on m. ■

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