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Weak-star Ergodic Problems and Enveloping Semigroups

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INTRODUCTION

The purpose of my talk is to present a unified view of three aspects of the theory of topological dynamical systems:

(1) the convergence of ergodic means;

(2) a relationship between minimal sets and ergodic measures;

(3) properties of associated enveloping semigroups.

In this context, it is natural to consider the convergence of functional and operator ergodic means in weak-* topologies; this approach essentially goes back to works of Kryloff and Bogoliouboff (1937) and Oxtoby (1952). In this talk, I consider only one-sided discrete systems (semicascades).

THREE DIRECTIONS IN ERGODIC THEORY

Ergodic theory began with the papers of Birkhoff and Neumann. Birkhoff's individual ergodic theorem asserts that the Cesàro means of functions $f \in L^1(\Omega, \mu)$ on a μ-measurable space Ω pointwise converge almost everywhere. Von Neumann's statistical theorem asserts the convergence of Cesàro means for powers of an isometric linear operator on L^2.

Von Neumann's theorem has given rise to the strong operator (SO) ergodic theory, which studies the convergence of various means of operator semigroups on Banach spaces.

Kryloff and Bogoliouboff proposed a third approach based on Birkhoff's theorem, which is related to the pointwise convergence of Cesàro means for continuous functions on a compact set Ω. In this connection, passing to operators in the dual space, we can consider the weak-* (W*O) convergence of operator means in $C^*(\Omega)$.

G.D. Birkhoff, *Proc. Acad. Sci. USA*, **17** (1931).

J. Neumann, *Proc.Nat. Acad. Sci. USA*, **18** (1932).

N. Kryloff and N. Bogoliouboff, *Ann. of Math.*, **38** (1937).

OBJECT OF STUDY

We consider a semicascade $(\Omega, \varphi) \doteq (\varphi^n, n \in \mathbb{N}_0)$, where φ is a continuous (not necessarily invertible) endomorphism of a compact metric space (Ω, ρ). The dynamics of the system may be very complex even for $\Omega = [0,1]$. We use the following notation:

$o(\omega) = \{\varphi^n \omega, n \geq 0\}$ and δ_ω are the Dirac measures for $\omega \in \Omega$;

$X = C(\Omega), \quad X^* = C^*(\Omega), \quad x \in X, \quad \mu \in X^*$;

$Ux = x \circ \varphi$ is the Koopman operator, $U \in \mathcal{L}(X)$;

$V = U^*, \quad V\mu = \mu \circ \varphi, \quad V \in \mathcal{L}(X^*)$;

$\mathcal{P}(\Omega)$ – Radon probability measures on Ω;

$\mathcal{P}i(\Omega) \subseteq \mathcal{P}(\Omega)$ denotes φ-invariant measures;

$\mathcal{P}e(\Omega)$ – ergodic measures μ_e;

m – minimal sets.

PROPERTIES OF DYNAMICS

We discuss the following dynamical properties of (Ω, φ):

(1) the closure of each orbit $\overline{o}(\omega)$ contains a unique minimal set;

(2) the supports of ergodic measures are minimal;

(3) each minimal set supports a unique ergodic measure;

(4) all transitive subsystems $(\overline{o}(\omega), \varphi)$, $\omega \in \Omega$, are uniquely ergodic;

(5) the equality $\overline{M(\Omega, \varphi)} = Z(\Omega, \varphi)$ holds, where $M(\Omega, \varphi)$ is the union of all m and $Z(\Omega, \varphi)$ is the minimal center of attraction, i.e., the closure of the union of supports of μ_e (in the general case, we have only the inclusion $\overline{M(\Omega, \varphi)} \subseteq Z(\Omega, \varphi)$);

(6) the proximality relation P on Ω (defined by setting $(\omega_1, \omega_2) \in \mathrm{P}$ if and only if $\inf\limits_{n \geq 0} \rho(\varphi^n \omega_1, \varphi^n \omega_2) = 0$) is transitive.

We have: $(1) + (2) + (3) \Leftrightarrow (4)$.

WEAK-STAR ERGODIC MEANS

A net $\{V_\alpha\} \subseteq \mathrm{co}\{V^n, n \geq 0\}$ of operators in $\mathcal{L}(X^*)$ is weak-star (W*O) ergodic if

$$(I-V)V_\alpha \overset{\mathrm{W^*O}}{\longrightarrow} 0: \quad (x, (I-V)V_\alpha \mu) \to 0,\ x \in X,\ \mu \in X^*.$$

Here $V_\alpha = U_\alpha^*$ and $U_\alpha \in \mathcal{L}(X)$; the net $\{U_\alpha\} \subseteq \mathrm{co}\,\{U^n, n \geq 0\}$ is referred to as weak-star ergodic as well. Thanks to the duality

$$(Ux, \mu) = (x, V\mu), \quad x \in X,\ \mu \in X^*,$$

the convergence $V_\alpha \mu \overset{w^*}{\longrightarrow} \overline{V}\mu$ in X^* is equivalent to the convergence $U_\alpha x \overset{w^*}{\longrightarrow} \overline{U}x$ ($\overline{U} = \overline{V}^*$) in X^{**}. If $V_\alpha \overset{\mathrm{W^*O}}{\longrightarrow} \overline{V}$ in $\mathcal{L}(X^*)$, then $\overline{V}^2 = \overline{V}$.

For ergodic sequences $\{U_n\} \subseteq \mathcal{L}(X)$, this convergence is equivalent to the pointwise convergence $U_n x \to \overline{x}$, $x \in X$, $\overline{x} \in B_1(\Omega) \subseteq X^{**}$, but in the general case, this is not true.

THE SEPARATION PRINCIPLE

We set $\mathrm{Fix}(U) = \{x \in X : Ux = x\}$ and $\mathrm{Fix}(V) = \{\mu \in X^* : V\mu = \mu\}$.

In the case of the SO (WO)-convergence of ergodic means, the well-known **separation principle** (Sine 1970, Nagel 1973, and Sato 1978) holds: **all ergodic nets converge if and only if** $\mathrm{Fix}(U)$ **separates** $\mathrm{Fix}(V)$.

What is the counterpart of this principle in the case of weak-* convergence?

R. Sine, *Proc. Amer. Math. Soc.*, **24** (1970).

R. Nagel, *Ann. Inst. Fourier, Grenoble*, **23** (1973).

R. Sato, *Tohoku Math. J.*, **30** (1978).

ERGODIC PROPERTIES

We discuss the following three basic questions concerning the weak-* ergodic properties of a semicascade (Ω, φ).

When do all ergodic nets of operators $V_\alpha \in \mathcal{L}(X^*)$ converge?

When do all ergodic sequences of operators $V_n \in \mathcal{L}(X^*)$ converge?

When does some ergodic sequence of operators $V_n \in \mathcal{L}(X^*)$ converge?

KRYLOFF–BOGOLIOUBOFF THEORY

Let

$$\overline{U}_n = \frac{1}{n}(I + U + ... + U^{n-1})$$

denote the Cesàro means for the Koopman operator $U = U_\varphi$ on $X = C(\Omega)$, and let

$$\overline{V}_n = \frac{1}{n}(I + V + ... + V^{n-1})$$

for the operator $V = U^*$ on $X^* = C^*(\Omega)$. The Kryloff–Bogoliouboff theory [1], developed by Oxtoby [2], considers the pointwise convergence

$$(\overline{U}_n x)(\omega) \to \overline{x}(\omega), \quad n \to \infty,$$

for $\omega \in \Omega$ and continuous functions $x \in X$.

1. N. Kryloff and N. Bogoliouboff, *Ann. of Math.*, **38** (1937).

2. J.C. Oxtoby, *Bull. Amer. Math. Soc.*, **58** (1952).

K–B CLASSIFICATION OF POINTS IN Ω

Quasi-regular points $(\omega \in \mathcal{Q})$: $(\overline{U}_n x)(\omega) \to \overline{x}(\omega) \quad \forall\, x \in X$.

This means the existence of a unique measure $\mu_\omega \in \mathcal{P}i(\Omega)$ such that $\overline{x}(\omega) = (x, \mu_\omega)$. In fact, $\overline{V}_n \delta_\omega \xrightarrow{w^*} \mu_\omega$ and **the measure μ_ω determines the asymptotic distribution of the orbit $o(\omega)$.**

Transitive points $(\omega \in \mathcal{Q}_T)$: quasi-regularity + $\mu_\omega \in \mathcal{P}e(\Omega)$.

Regular points $(\omega \in \mathcal{R})$: transitivity + $\omega \in \operatorname{supp} \mu_\omega$.

Theorem [1]. *The sets $\mathcal{Q} \supseteq \mathcal{Q}_T \supseteq \mathcal{R}$ are Borel, and $\mu(\mathcal{R}) = 1\ \forall \mu \in \mathcal{P}i(\Omega)$, i.e.,* $\overline{R} \supseteq Z(\Omega, \varphi)$.

To the ergodic measures μ there correspond the Borel **ergodic sets**

$$\mathcal{R}_\mu = \{\omega \in \mathcal{R} : \mu_\omega = \mu\},$$

which form a partition of the set $\mathcal{R}$ of regular points.

1. N. Kryloff and N. Bogoliouboff, *Ann. of Math.*, **38** (1937).

2. J.C. Oxtoby, *Bull. Amer. Math. Soc.*, **58** (1952).

ORBITS WITH HISTORIC BEHAVIOR

We say that an orbit $o(\omega)$ exhibits historic behavior (preserves information about the past) if the point $\omega \in \Omega$ is not quasi-regular. To such orbits the following well-known problem is related.

Takens' last problem [1]: *Under what conditions can such orbits occur persistently*?

In [2] an important step toward solving this problem was made; namely, a structurally stable family of dynamical systems on a smooth manifold for which the set of non-quasi-regular points has nonempty interior was constructed.

1. F. Takens, *Nonlinearity*, **21** (2008).

2. I.S. Labouriau and A.P. Rodrigues, *Nonlinearity*, **30** (2017).

THE WEAK-STAR SEPARATION PRINCIPLE

Let us set

$$X_0 = \{x \in X : U_\alpha x \xrightarrow{w^*} z \in X^{**}\},$$

then $z = \overline{U}x$, where $\overline{U} \in \mathcal{L}(X_0, X^{**})$ and $z \in \mathrm{Fix}(U^{**})$.

Iwanik's criterion [1]: The Cesàro means $\overline{U}_n x$ for $Ux = x \circ \varphi$, $x \in X$, pointwise converge to $\overline{U}x$ (all $\omega \in \Omega$ are quasi-regular!) if and only if $\overline{U}X_0$ separates $\mathrm{Fix}(V)$**.** This assertion can be generalized.

Theorem [2]. *For any* W*O*-ergodic net* $\{U_\alpha\} \subseteq \mathcal{L}(X)$, *the following conditions are equivalent:*

(a) $\overline{U}X_0$ *separates* $\mathrm{Fix}(V)$;

(b) $X_0 = X$.

Thus, a W*O**-ergodic net** $\{U_\alpha\} \subseteq \mathcal{L}(X)$ **converges if and only if** $\overline{U}X_0$ **separates** $\mathrm{Fix}(V)$**.**

1. A. Iwanik, *Bull. Acad. Polon. Sci., Ser. Sci. Math.*, **29** (1981).

2. A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

ENVELOPING SEMIGROUPS

We set: $E_0(\Omega,\varphi)=\{\varphi^n,\, n\in\mathbb{N}_0\}$, $E_0(\Omega,\varphi)\subseteq C(\Omega,\Omega)$;

$K_0(\Omega,\varphi)=\{V^n,\, n\in\mathbb{N}_0\}$, $K_0(\Omega,\varphi)\subseteq \mathcal{L}(X^*)$;

$G_0(\Omega,\varphi)=\mathrm{co}\, K_0(\Omega,\varphi)$, $G_0(\Omega,\varphi)\subseteq \mathcal{L}(X^*)$.

The Ellis semigroup $E(\Omega,\varphi)=\overline{E_0(\Omega,\varphi)}$ in the product topology of Ω^Ω;

The Köhler semigroup $K(\Omega,\varphi)=\overline{K_0(\Omega,\varphi)}$ in the W*O-topology of $\mathcal{L}(X^*)$;

The semigroup $G(\Omega,\varphi)=\overline{G_0(\Omega,\varphi)}$ in the W*O-topology of $\mathcal{L}(X^*)$.

The operator V generates the semicascade $(\mathcal{P},V)$ on the w^*-compact set $\mathcal{P}=\mathcal{P}(\Omega)$ in X^*, and $K(\Omega,\varphi)\simeq E(\mathcal{P},V)$. Moreover,

$$G(\Omega,\varphi)=\overline{\mathrm{co}}\, K(\Omega,\varphi),\quad G_0G=GG_0,\quad \|T\|_{X^*}=1 \text{ for } T\in G(\Omega,\varphi).$$

The right-topological semigroups $E(\Omega,\varphi)$, $K(\Omega,\varphi)$, and $G(\Omega,\varphi)$ are compact. In fact, $G(\Omega,\varphi)\simeq E(\mathcal{P},S)$ with Abelian polynomial semigroup $S=\mathrm{co}\{t^n,\, n\geq 0\}$.

The semigroup $G(\Omega,\varphi)$ is the key object responsible for the ergodic properties of (Ω,φ).

R. Ellis, *Lectures on Topological Dynamics*, 1969.

A. Köhler, *Proc. Roy. Irish. Acad.*, **95**A (1995).

A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

THE KERNEL OF THE SEMIGROUP $G(\Omega,\varphi)$

The kernel of any semigroup is the intersection of all two-sided ideals (= the union of all minimal left (right) ideals).

We have: $\mathrm{Ker}\, G = \{Q \in G : VQ = Q\} = \{Q \in G : \mathrm{Im}\, Q = \mathrm{Fix}(V)\}$,

$$GQ = Q, \quad QG_0 = Q, \quad Q^2 = Q, \quad \|Q\|_{X^*} = 1.$$

The convex W*O-compact set $\mathrm{Ker}\, G \subset \mathcal{L}(X^*)$ of operators was first considered by Lloyd [1]. Apparently, it is the algebraic-geometric properties of this object which are responsible for the W*O-ergodic properties of the semicascade (Ω,φ) [2].

Lemma 1. $\mathrm{Ker}\, G$ *equals the union of all one-element left ideals and is a unique minimal right ideal in the semigroup* $G(\Omega,\varphi)$.

Let $\mathrm{cl}\,\{V_\alpha\}$ be the set of cluster points of any operator net $\{V_\alpha\} \subseteq G_0(\Omega,\varphi)$.

Lemma 2. *An operator net* $\{V_\alpha\} \subseteq G_0$ *is ergodic* $\Leftrightarrow \mathrm{cl}\,\{V_\alpha\} \subseteq \mathrm{Ker}\, G$.

Theorem. *All ergodic nets* $\{V_\alpha\} \subset \mathcal{L}(X^*)$ *converge* $\Leftrightarrow$ $\mathrm{Ker}\, G \ni$ *single* Q – *zero of* $G(\Omega,\varphi)$.

1. S.P. Lloyd, *Proc. Amer. Math. Soc.*, **56** (1976).

2. A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

ON WEAK-STAR CONVERGENCE

The relation $VQ = Q$ for $Q \in \operatorname{Ker} G$ imply the following assertion.

Corollary. *Each element $Q \in \operatorname{Ker} G$ is a limit of an ergodic net $\{V_\alpha\} \subset \mathcal{L}(X^*)$; therefore, W^*O-converging ergodic nets always exist.*

Question. *Under what conditions does there exist a W^*O-convergent ergodic sequence?*

The difference between strong (weak) and weak-* ergodic theories: According to the separation principle, all SO (WO)-ergodic nets converge or diverge simultaneously.

A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

REGULAR ELEMENTS OF THE ELLIS SEMIGROUP – 1

In what follows:

Σ_b and Σ_u – Borel and universally measurable sets in Ω;

Π_b and Π_u – Borel and universally measurable transformations of Ω.

Definition. *A transformation* $p \in E(\Omega, \varphi)$ *is regular if*

$$p \in \Pi_u, \qquad \mu(p^{-1}h) = \mu(h) \ \forall \ \mu \in \mathcal{P}e(\Omega), \ h \in \Sigma_u.$$

Let $E_r(\Omega, \varphi)$ be the set of all regular p, and let $E_b(\Omega, \varphi) \subseteq \Pi_b$ be the minimal sequentially closed subset of $E(\Omega, \varphi)$ containing $E_0(\Omega, \varphi)$. A transfinite procedure (similar to the scheme of construction of the Baire classes) proves the (generally, proper) inclusion $E_b(\Omega, \varphi) \subset E_r(\Omega, \varphi)$.

A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

REGULAR ELEMENTS OF THE ELLIS SEMIGROUP – 2

We consider the following properties of the semicascade (Ω, φ):

(KR) – the kernel $\operatorname{Ker} E(\Omega, \varphi)$ contains a regular element;

(single m in $\overline{o}$) – the closure of each orbit $\overline{o}(\omega)$ contains a unique m;

($\operatorname{supp} \mu_e = m$) – the supports of ergodic measures are minimal;

(single μ_e on m) – each minimal set supports a unique ergodic measure;

(AES) – all ergodic sequences of operators $V_n \in \mathcal{L}(X^*)$ converge.

Theorem. *The following implications hold:*

(KR) + (single m in $\overline{o}$) $\Rightarrow$ ($\operatorname{supp} \mu_e = m$);

(KR) + (single m in $\overline{o}$)+ (single μ_e on m) $\Rightarrow$ (AES);

(KR) $\Rightarrow$ $\overline{M(\Omega, \varphi)} = Z(\Omega, \varphi)$.

Remark. The equality $\overline{M(\Omega, \varphi)} = Z(\Omega, \varphi)$ does not imply ($\operatorname{supp} \mu_e = m$).

A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

TAME SEMICASCADES

Tame dynamical systems were introduced (under a different name) by Köhler [1] and renamed by Glasner [2].

Definition [1]. A semicascade (Ω, φ) is said to be tame if, for any $x \in C(\Omega)$ and any subsequence $\{n(k)\} \subseteq \mathbb{N}_0$, we have

$$\inf_a \left\| \sum_{k=0}^{\infty} a_k x_{n(k)} \right\|_{C(\Omega)} = 0,$$

where $x_{n(k)} = x \circ \varphi^{n(k)}$, the $a \in l_1$ are finite, and $\sum_{k=0}^{\infty} |a_k| = 1$.

In fact, this means that the shifts of any continuous function are "almost linearly dependent". Tame systems have a number of remarkable topological dynamical [2] and ergodic properties. We denote the class of tame semicascades by $\mathcal{D}_{\mathrm{tm}}$.

Key property: $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{tm}} \overset{[2,3]}{\Leftrightarrow} G(\Omega, \varphi)$ is a Fréchet–Urysohn topological space.

1. A. Köhler, *Proc. Roy. Irish. Acad.*, **95**A (1995).

2. E. Glasner, *Topology Appl.*, **154** (2007).

3. H. Kreidler, arXiv: 1703.05014v2.

THE MAIN ASSERTIONS: THE GENERAL CASE

Consider the following properties of a semicascade (Ω, φ):

(UE $(\overline{o}, \varphi)$) – all transitive subsystems $(\overline{o}(\omega), \varphi)$, $\omega \in \Omega$, are uniquely ergodic;

(AEN) – all ergodic nets of operators $V_\alpha \in \mathcal{L}(X^*)$ converge;

(AES) – all ergodic sequences of operators $V_n \in \mathcal{L}(X^*)$ converge;

(SEBS) – there is a convergent ergodic sequence $\{V_n\}$ for which the individual ergodic theorem is valid.

Theorem. *The following implications hold:*

(AEN) $\overset{[3]}{\Rightarrow}$ (UE $(\overline{o}, \varphi)$) $\overset{[2]}{\Rightarrow}$ (AES) $\overset{\text{trivial}}{\Rightarrow}$ (SEBS) $\overset{[1]+}{\Rightarrow}$ (single μ_e on m).

1. J.C. Oxtoby, *Bull. Amer. Math. Soc.*, **58** (1952).

2. A.V. Romanov, *Izvestiya: Mathematics*, **75** (2011).

3. H. Kreidler, arXiv: 1703.05014v2.

THE MAIN ASSERTIONS: TAME SYSTEMS – 1

The semigroup $G(\Omega, \varphi)$ is a Fréchet–Urysohn topological space in this case.

Theorem 1 ([1+2]). *For* $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{tm}}$, *each ergodic net of operators* $V_\alpha \in \mathcal{L}(X^*)$ *contains a convergent ergodic sequence* $\{V_{\alpha(k)}\}$.

Remark. *If* $\{V_\alpha\} = \{V_n\}$, *then* $\{V_{n(k)}\}$ *is a subsequence.*

Corollary [1]. *For any tame system, the Cesàro means contain a convergent subsequence.*

Let (SES) denote the existence of a convergent ergodic sequence of operators $V_n \in \mathcal{L}(X^*)$.

Theorem 2. *For* $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{tm}}$, *condition* (SES) *holds* [1] *and, moreover:*

(AEN) $\overset{[1]}{\Leftrightarrow}$ (AES) $\overset{[1,2]}{\Leftrightarrow}$ (single μ_e on m); (single μ_e on m) $\overset{[2]}{\Rightarrow}$ (UE $(\overline{o}, \varphi)$).

1. A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

2. H. Kreidler, arXiv: 1703.05014v2.

THE MAIN ASSERTIONS: TAME SYSTEMS – 2

On the other hand, (SES) $\not\Rightarrow$ (AES): a tame semicascade (Ω, φ) was constructed for which the Cesàro means converge but (AES) does not hold [1].

Theorem [1+2]. *If* $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{tm}}$*, then for the* W*O*-convergence of an ergodic net of operators* $V_\alpha \in \mathcal{L}(X^*)$ *it is sufficient that the functional nets* $U_\alpha x$ *pointwise converge on* Ω *for all* $x \in X$.

Theorem [2]. *If* $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{tm}}$*, then* $\overline{M(\Omega, \varphi)} = Z(\Omega, \varphi)$.

1. H. Kreidler, arXiv: 1703.05014v2.

2. A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

THE PROXIMALITY RELATION

Let (P) denote the transitivity of the proximality relation in a system (Ω, φ).

Theorem. *For any system* (Ω, φ), (P) $\Rightarrow$ (single m in $\bar{o}$).

Combining this result with the preceding ones, we obtain the following assertion.

Corollary. *For any tame system* (Ω, φ), (P) $\Rightarrow$ (AEN) and, in particular, (P) implies the convergence of Cesàro means.

A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

ASYMPTOTIC DISTRIBUTION OF ORBITS – 1

Let $\mathcal{V} = \{V_n\}$ be any ergodic sequence of operators in $\mathcal{L}(X^*)$. We can speak about $\mathcal{V}$-quasi-regular, $\mathcal{V}$-transitive, and $\mathcal{V}$-regular points $\omega \in \Omega$ and about invariant measures μ_ω determining the asymptotic $\mathcal{V}$-distribution of the orbits $o(\omega)$.

Lemma. *If $Q \in \mathrm{ex\,Ker}\, G$, then $Q : D(\Omega) \to \mathcal{P}e(\Omega)$, where $D(\Omega)$ is the set of Dirac measures on Ω.*

Let $Q \in \mathrm{ex\,Ker}\, G$. The convergence $V_\alpha \xrightarrow{\mathrm{W^*O}} Q$ of an operator ergodic net $\{V_\alpha\} \subset \mathcal{L}(X^*)$ implies the convergence $V_\alpha \delta_\omega \xrightarrow{w^*} Q\delta_\omega$ and the ergodicity of the measures $Q\delta_\omega$. Since the w^*-compact space $\mathcal{P}(\Omega)$ is metrizable, it follows that, for any $\omega \in \Omega$, there exists an ergodic subsequence for which $V_{\alpha(n)}\delta_\omega \xrightarrow{w^*} Q\delta_\omega$.

Corollary. *Each point $\omega \in \Omega$ is transitive with respect to some (depending of ω) ergodic sequence $\mathcal{V}_\omega$, i.e., there exists an ergodic measure μ_ω determining the asymptotic $\mathcal{V}_\omega$-distribution of the orbit $o(\omega)$.*

A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

ASYMPTOTIC DISTRIBUTION OF ORBITS – 2

In the tame case, we can say more.

Theorem [1+]. *If* $(\Omega,\varphi) \in \mathcal{D}_{\mathrm{tm}}$ *and* $Q \in \operatorname{ex}\operatorname{Ker} G$, *then there exists an ergodic sequence of operators* V_n *such that* $V_n \xrightarrow{\mathrm{W^*O}} Q$.

Thus, for a tame semicascade (Ω,φ), all points $\omega \in \Omega$ are transitive with respect to some ergodic sequence $\mathcal{V}_\varphi$, and the asymptotic $\mathcal{V}_\varphi$-distributions of all orbits are determined by ergodic measures.

1. A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

MORE ON TAME SYSTEMS

There are many equivalent definitions of tame systems, in particular:

(1) the topological space $E(\Omega, \varphi)$ is Fréchet–Urysohn;

(2) the topological space $E(\Omega, \varphi)$ is sequential;

(3) $\operatorname{card} E(\Omega, \varphi) \leq \mathfrak{c}$;

(4) $\operatorname{card} G(\Omega, \varphi) \leq \mathfrak{c}$;

(5) the topological space $G(\Omega, \varphi)$ is Fréchet–Urysohn;

(6) $E(\Omega, \varphi) \subseteq \Pi_b$;

(7) $E(\Omega, \varphi) = \{\varphi^n, n \geq 0\}_s$, where $(\cdot)_s$ denotes sequential limits;

(8) $\forall\, x \in X$ the orbit $\{x \circ \varphi^n, n \geq 0\}$ is RSK in the topology of $\mathbb{R}^\Omega$.

E. Glasner, *Topology Appl.*, **154** (2007).

A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016) + *unpublished.*

H. Kreidler, arXiv: 1703.05014v2.

ORDINARY AND INJECTIVE SYSTEMS – 1

The correspondence $V \to \varphi$ generates the continuous algebraic epimorphism [1]

$$K(\Omega,\varphi) \xrightarrow{\pi} E(\Omega,\varphi), \quad \pi : P \to p;$$

moreover, $P\delta_\omega = \delta_{p\omega}$, $\omega \in \Omega$. Consider two important classes of dynamical systems.

The class $\mathcal{D}_{\mathrm{or}}$ of ordinary systems, for which the semigroup $E(\Omega,\varphi)$ is metrizable. We have $(\Omega,\varphi) \in \mathcal{D}_{\mathrm{or}} \overset{[4]}{\Leftrightarrow} G(\Omega,\varphi)$ is metrizable.

The class $\mathcal{D}_{\mathrm{in}}$ of injective ([2]) systems, for which π is a homeomorphism [1]; equivalently, all operators $P \in K(\Omega,\varphi)$ are determined by their values at the δ_ω.

The following inclusions hold: $\mathcal{D}_{\mathrm{or}} \subset \mathcal{D}_{\mathrm{tm}} \subset \mathcal{D}_{\mathrm{in}}$.

1. A. Köhler, *Proc. Roy. Irish. Acad.*, **95**A (1995).

2. E. Glasner, *Topology Appl.*, **154** (2007).

3. E. Glasner, M. Megrelishvili and V. V. Uspenskij, *Israel J. Math.*, **164** (2008).

4. A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

ORDINARY AND INJECTIVE SYSTEMS – 2

For injective systems, the action of $E(\Omega,\varphi)$ is naturally transferred from Ω to $\mathcal{P}(\Omega)$, which makes it possible to essentially identify the Ellis and Köhler semigroups.

Since $(\Omega,\varphi)\in\mathcal{D}_{\rm or}\Leftrightarrow G(\Omega,\varphi)$ is metrizable, the following assertion is valid.

Theorem. *For an ordinary semicascade, each ergodic net contains a convergent ergodic subsequence.*

A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016).

SOME EXAMPLES

Any homeomorphism of I or $\mathbb{S}^1$ generates an ordinary system; each weakly almost periodic semicascade belongs to $\mathcal{D}_{\mathrm{or}}$ [1, 2].

EXAMPLES [1]. *The semicascade generated by the Bernoulli shift on the Cantor space* $\Omega = \{0,1\}^{\mathbb{N}_0}$ *is injective but not tame. The affine cascade*

$$(\mathbb{T}^2, \varphi), \quad \varphi(\omega_1, \omega_2) = (\omega_1 + \theta, \omega_2 + \omega_1) \pmod 1, \quad \theta \in \mathbb{R} \setminus \mathbb{Q},$$

is not injective.

There are exist subshifts dividing the classes of ordinary and tame cascades [3].

1. E. Glasner, *Topology Appl.*, **154** (2007).

2. E. Glasner and M. Megrelishvili, *Colloq. Math.*, **104** (2006).

3. M. Megrelishvili, *Privately communication.*

MORE ON CLASSES OF DYNAMICAL SYSTEMS

Let us distinguish two more classes of semicascades, which are of interest for weak-* ergodic theory:

the class $\mathcal{D}_{\mathrm{pw}}$, for which the operators $T \in G(\Omega, \varphi)$ are determined by their values at the δ_ω, and the class $\mathcal{D}_{\mathrm{um}}$, for which $E(\Omega, \varphi) \subseteq \Pi_u$. The following inclusions hold:

$$\mathcal{D}_{\mathrm{tm}} \subseteq \mathcal{D}_{\mathrm{pw}} \subseteq \mathcal{D}_{\mathrm{in}}, \quad \mathcal{D}_{\mathrm{tm}} \subseteq \mathcal{D}_{\mathrm{um}}.$$

The question of whether these inclusions are proper is still open.

However, it is consistent with ZFC, then $\mathcal{D}_{\mathrm{tm}} = \mathcal{D}_{\mathrm{um}}$ and $\mathcal{D}_{\mathrm{um}} \subset \mathcal{D}_{\mathrm{in}}$ [2].

To the class $\mathcal{D}_{\mathrm{pw}}$ the following useful theorem, which was stated above for tame systems, can be extended.

Theorem. *If $(\Omega, \varphi) \in \mathcal{D}_{\mathrm{pw}}$, then for the* W*O*-convergence of an ergodic net of operators $V_\alpha \in \mathcal{L}(X^*)$ it is sufficient that the functional nets $U_\alpha x$ pointwise converge on Ω for all $x \in X$.*

1. A.V. Romanov, *Ergod. Theory and Dynam. Syst.*, **36** (2016) + *unpublished.*

2. A.V. Romanov, *Int. J. Math. Anal.*, **11** (2017).

WHAT ELSE CAN BE DONE? – 1

The global goal: the common weak-star view on ergodic problems.

The local goal: fill the theory with nontrivial examples!

Examine the following simple discrete dynamical systems with complex orbit structure from the point of view of what was said above:

(a) epimorphisms of $[0, 1]$ –

$\varphi x = nx \pmod 1$

the logistic epimorphisms $\varphi x = \lambda x(1 - x)$, $0 < \lambda \leq 4$,

the tent transformation;

(b) the Bernoulli shift and its subshifts;

(c) the Baker transformation;

(d) affine cascades on tori and so on (the list can be extended).

WHAT ELSE CAN BE DONE? – 2

For "popular" discrete dynamical systems, investigate the following questions:

(1) the convergence of Cesàro means;

(2) the convergence of some ergodic sequences;

(3) the convergence of all ergodic sequences;

(4) the convergence of all ergodic nets.

Begin to classify similar systems in terms of associated enveloping semigroups.

In particular, it is known that the semicascades of type (a) are not tame [3, 4]. The semicascades $(\mathbb{S}^{n-1}, T)$, $T \in LO(\mathbb{R}^n)$, are tame [2].

It would be interesting to comprehend the deep results of [1] on the weak-* convergence of Cesàro means for cascades on tori from the point of view of enveloping semigroups.

1. H. Furstenberg, *Amer. J. Math.*, **83** (1961).

2. E. Glasner, *Topology Appl.*, **154** (2007).

3. E. Glasner and M. Megrelishvili, *Submitted*.

4. Vladimir Lebedev, *Privately communication*.


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