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Inertial manifolds of finite-dimensional dynamical systems

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In the case of a semilinear parabolic equation with infinite-dimensional phase space X, the existence of an inertial manifold implies that the final dynamics of the system can be described by a differential equation in $\mathbb{R}^n$ [1]. We show that inertial manifolds are also helpful in the case of finite-dimensional X. For example, under certain circumstances, the existence of a two-dimensional inertial manifold makes it possible to prove the existence of an asymptotically orbitally stable limit cycle without using bifurcation theory [2] or complicated geometric constructions related to fixed point theorems [3]. Based on the sharp spectral gap condition [4], we prove the existence of stable limit cycles for certain well-known models in mathematical biology.

1. G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer, New York, 2002.
2. J.E. Marsden and M. McCracen, The Hopf Bifurcation and its Applications, 1976.
3. V.A. Pliss, Nonlocal Problems of the Theory of Oscillations, 1966.
4. M. Miklavcic, *J. Dyn. and Differ. Equat.*, **3**:3 (1991).

Problem statement

We consider systems of equations of the form

$$\dot{x} = -Ax + F(x), \quad x \in \mathbb{R}^n, \quad n \geq 3, \qquad (*)$$

where A is a symmetric $n\times n$ matrix with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and $F \in C^{1+\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$. If

$$\|F'(x)\|_2 \leq K, \quad x \in \mathbb{R}^n,$$

where $\|\cdot\|_2$ is the spectral matrix norm, then system $(*)$ generates a smooth phase semiflow $\{S_t, t \geq 0\}$ in $\mathbb{R}^n$.

We set $f(x) \doteq -Ax + F(x)$; by $\|\cdot\|$ and $(\cdot,\cdot)$ we denote, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$.

A domain $D \subseteq \mathbb{R}^n$ is (strictly) ***positively invariant*** if the vector field f on ∂D is **directed inside** D and, therefore, $S_t D \subseteq D$, $t > 0$. A set $\Lambda \subseteq \mathbb{R}^n$ is ***invariant*** if $S_t \Lambda = \Lambda$, $t > 0$.

A **cycle** is a **closed orbit**.

OBJECTIVES:

(max) prove the existence of an asymptotically (orbitally) stable limit sycle;
(min–1) reduce the study of the final dynamics in $\mathbb{R}^2$;
(min–2) in the case $n = 3$, determine the two-dimensional Cartesian structure of cycles (if they exist).

Intertial manifolds

An inertial manifold is a smooth invariant m-dimensional surface $H_m \subset \mathbb{R}^n$ $(m<n)$ exponentionally attracting all orbits $x(t)$ as $t \to +\infty$.

Let P and Q be the spectral orthoprojections of the matrix A onto the subspaces X_m and X_{n-m} corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_m$ and $\lambda_{m+1},\ldots,\lambda_n$, respectively. As is known [1], under the ***sharp spectral gap condition*** $\lambda_{m+1}-\lambda_m > 2K$, an inertial manifold can be constructed as a graph

$$H_m = \{x \in \mathbb{R}^n :\ x = \psi(u)\}, \quad u \in X_m,$$

where $\psi(u) = u + \sigma(u)$ for a $C^{1+\varepsilon}$ function $\sigma: X_m \to X_{n-m}$. Moreover, for each orbit $x(t)$, there exists an orbit $\bar{x}(t) \subset H_m$ such that

$$\left\| x(0) - \bar{x}(0) \right\| \le M_1 \left\| Qx(0) - \sigma(Px(0)) \right\|, \qquad (1)$$

$$\left\| x(t) - \bar{x}(t) \right\| \le M_2 e^{-\gamma t} \left\| x(0) - \bar{x}(0) \right\|$$

for $t \ge 1$, where $\gamma, M_1, M_2 = \text{const} > 0$.

1. S. Zelik, *Proc. Roy. Soc. Edinburgh, Ser. A*, **144**:6 (2014).

Intertial manifolds. 1

Any inertial manifold contains all invariant compact sets (including the fixed points and cycles) of the dynamical system.

The reduction principle [1]: the invariant compact sets Λ of system $(*)$ and $P\Lambda$ of system

$$\dot{u} = -Au + PF(\psi(u)), \quad u = Px, \qquad (**)$$

in $X_m \simeq \mathbb{R}^m$ **are asymptotically stable or not asymptotically stable simultaneously.** The *inertial form* $(**)$ is topologically equivalent to the restriction of the initial system $(*)$ to H_m.

The existence of an inertial manifold makes it possible to reduce analyzing final modes to solving a similar problem for ODEs in $\mathbb{R}^m, m < n$**.**

1. A.V. Romanov, *Izv. Math.*, **43**:1 (1994).

Intertial manifolds. 2

If a nonlinear function $F:\mathbb{R}^n \to \mathbb{R}^n$ is only Lipschitz continuous, i.e., ,

$$\|F(x)-F(x')\| \le K, \quad x, x' \in \mathbb{R}^n,$$

and $\lambda_{m+1} - \lambda_m > 2K$, then there exists a *Lipschitz* m-dimensional inertial manifold $H_m \subset \mathbb{R}^n$ [1, 2]**.** An inertial manifold H_m also exists in the case of a ***nonpositive symmetric matrix*** A, provided that $\lambda_{m+1} - \lambda_m > 2K$ and $\lambda_{m+1} > K$.

REMARK. In the presence of a two-dimensional inertial manifold, any cycle (if it exists) has the form of a graph over a plane in $\mathbb{R}^n$.

1. M. Miklavcic, *J. Dyn. and Differ. Equat.*, **3**:3 (1991).

2. A.V. Romanov, *Izv. Math.*, **43**:1 (1994).

Cycles: The spectral gap method

THEOREM (main). *Suppose that system* $(*)$ *satisfies the following conditions*:

(i) *there exists a positively invariant bounded domain* $D \subset \mathbb{R}^n$ *and the function* $F : D \to \mathbb{R}^n$ *is real analytic*;

(ii) *there exists a unique fixed point* $x_s \in D$;

(iii) $\lambda_3 - \lambda_2 > 2K$.

Then the following assertions hold.

(a) *Either* x_s *attracts the entire domain* D, *or* D *contains at least one cycle.*

(b) *If, in addition,*

(iv) *the point* x_s *is asymptotically unstable,*

then D *contains at least one asymptotically stable cycle.*

Using (iii), we reduce the final dynamics $(*)$ to an inertial manifold $H_2 \simeq \mathbb{R}^2$ containing x_s, and assertion (a) follows from Poincaré–Bendixson theory. Since $\mathrm{ind}(x_s) = 1$, it follows that, in case (b), the Jacobian matrix $-A + F'(x_s)$ has precisely two complex conjugate eigenvalues in the domain $\operatorname{Re} z > 0$ and the point Px_s is an unstable focus of system $(**)$ in the plane X_2. The real analyticity of F implies that D contains at most finitely many cycles of system $(*)$. According to Poincaré–Bendixson theory, the inertial form $(**)$ has $l \geq 1$ embedded cycles in the domain $PD \subset X_2$, and at least one of these cycles, say Γ, is asymptotically stable. Then $\psi\Gamma$ is an asymptotically stable limit cycle of the initial system $(*)$.

Cycles: The cone method

In fact, Russell Smith applied inertial manifolds to study limit cycles for ODEs and PDEs as early as in [1984–1994]. To such manifolds he initially gave the name of amenable manifolds [1] and used the following **cone condition** rather than the spectral gap condition:

there exist $\lambda, \varepsilon > 0$ *such that any solution* $x(t), x'(t)$ *of the equation* $\dot{x} = f(x)$ *in* $\mathbb{R}^n$ *satisfies the relation*

$$\frac{d}{dt}V(x(t)-x'(t)) + 2\lambda V(x(t)-x'(t)) \le -\varepsilon \|x(t)-x'(t)\|^2, \qquad (1)$$

where $V(x)=(Jx,x)$ *for some nonsingular symmetric matrix* J with m *negative and* $n-m$ *positive eigenvalues* .

Inequality (1) follows from the spectral gap condition, and it also implies the existence of an m-dimensional inertial manifold [1]. For abstract semilinear parabolic equations in Hilbert space $(X, \|\cdot\|)$, a similar condition was considered in the case

$$V(x) = \|Qx\|^2 - \xi^2 \|Px\|^2 \quad (\xi > 0),$$

when P and Q are orthoprojections in X , $P+Q=\mathrm{Id}$, and $\dim PX = m$ [2–4].

The cone condition for ODEs is weaker than the spectral gap condition!

1. R.A. Smith, *Proc. London Math. Soc.* (3), **48**:2 (1984).
3. R. Temam, Infinite-dimensional Dynamical Systems in Mechanics and Physics, 1988.
3. J.C. Robinson, *Dyn. Sys. Appl*. **2**:3 (1993).
4. A.V. Romanov, *Izv. Math*., **43**:1 (1994).

Cycles: The cone method. 1

For the limit cycles of the equation

$$\dot{x} = -Ax + F(x), \quad x \in \mathbb{R}^n, \quad n \ge 3,$$

with an ***arbitrary matrix*** A, a statement similar to the main theorem was proved in [1] on the basis of the cone condition (1). Condition (1) is derived from the existence of a $\lambda > 0$ with the following properties:

(i) $\lambda + i\omega \in \rho(A)$ for $\omega \in \mathbb{R}$, and the matrix A has precisely two eigenvalues with $\operatorname{Re} z < \lambda$;

(ii) $K\left\|(\lambda + i\omega - A)^{-1}\right\|_2 < 1$ for $\omega \in \mathbb{R}$ and $\left\|F'(x)\right\|_2 \le K$.

In fact, these are M. Miklavcic's [2] conditions for the existence of an inertial manifold in a separable Hilbert space for a sectorial linear operator A.

Suppose that F has a form typical of ***feedback control systems***, i.e.,

$$F(x) = B\Phi(Cx), \tag{2}$$

where $\Phi \in C^1(\mathbb{R}^s, \mathbb{R}^r)$, B is a constant $n \times r$ matrix, and C is a constant $s \times n$ matrix. Then (ii) can be replaced by

(iii) $K\left\|\chi(\lambda + i\omega)\right\|_2 < 1$ for $\omega \in \mathbb{R}$, $\left\|\Phi'(x)\right\|_2 \le K$, $\chi(z) = C(zI - A)^{-1}B.$

1. R.A. Smith, *J. Differ. Eq.*, **69**:2 (1987).
2. M. Miklavcic, *J. Dyn. and Differ. Equat.*, **3**:3 (1991).

Comparison of the two methods

In the case of $A = A^*$, the methods are equivalent.

The cone method is more general, because it does not require the matrix A to be symmetric and leaves freedom in the choice of a representation of the nonlinear part of the system in the form (2). At the same time, the application of this method may involve serious technical difficulties.

The spectral gap method assumes the existence of a natural self-adjoint linear component of the vector field of the system. This narrows its applicability domain. The method has the following advantages:

- transparent formulation;
- relative simplicity of application: it is much easier to check the spectral gap condition than the cone condition.

Below we illustrate the spectral gap method by two examples.

Satellite control

The spectral gap method was applied in [1] to study the dynamics of a satellite flying around a celestial object of small mass.

Objective: choose numerical control parameters and a control function ensuring the existence of a steady periodic (in appropriate coordinates) motion. As a generalization of results of [1], we consider the system of equations

$$\begin{aligned} \dot{x}_1 &= -\mu_1 x_1 + g(x_3) \\ \dot{x}_2 &= -\mu_2 x_2 + x_1 \\ \dot{x}_3 &= -\mu_3 x_3 + x_2 \end{aligned} \qquad (1)$$

with control parameters $\mu_1, \mu_2, \mu_3 > 0$ and a control function $g \in C^1(\mathbb{R})$ (depending on the parameters!). Here (x_1, x_2, x_3) are suitable variables determining the dynamics of the satellite.

A similar system was considered in [2] from a different point of view.

Suppose that

$$g(x_3) > 0 \text{ and } -1 \le g'(x_3) < 0 \text{ for } x_3 \in \mathbb{R}.$$

Then system (1) takes the form $(*)$ for

$$A = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \qquad F(x) = \begin{pmatrix} g(x_3) \\ x_1 \\ x_2 \end{pmatrix}.$$

1. I.A. Galiullin and L.A. Kondrateva, *Cosmonautics and rocket Engineering*, №4 (2011), in Russian.
2. R.A. Smith, *J. Differ. Eq.*, **69**:2 (1987).

Satellite control. 1

For the given system, the following assertions hold.

(1) For each $\beta > 1$, the domain

$$D_\beta = \{ \frac{g(\theta)}{\beta\mu_1} < x_1 < \frac{\beta g(0)}{\mu_1}, \quad \frac{g(\theta)}{\beta^2\mu_1\mu_2} < x_2 < \frac{\beta^2 g(0)}{\mu_1\mu_2}, \quad \frac{g(\theta)}{\beta^3\mu_1\mu_2\mu_3} < x_3 < \frac{\beta^3 g(0)}{\mu_1\mu_2\mu_3} \},$$

where $\theta = \frac{\beta^3 g(0)}{\mu_1\mu_2\mu_3}$, is positively invariant [1].

(2) Each domain D_β contains a unique fixed point $p = \left\{ \frac{g(\nu)}{\mu_1}, \frac{g(\nu)}{\mu_1\mu_2}, \nu \right\}$, where $\nu > 0$ is a unique solution of the equation $\frac{g(\nu)}{\nu} = \mu_1\mu_2\mu_3$.

(3) The following relations hold:

$$F'(x) = \begin{pmatrix} 0 & 0 & g'(x_3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F'(x)\cdot\left(F'(x)\right)^* = \operatorname{diag}((g'(x_3))^2, 1, 1), \quad \left\|F'(x)\right\|_2 = 1, \quad x \in \mathbb{R}^3.$$

1. R.A. Smith, *J. Differ. Eq.*, **69**:2 (1987).

Satellite control. 2

Let $\lambda_1 \le \lambda_2 \le \lambda_3$ be the ordered parameters μ_1, μ_2, μ_3.
Since $K = 1$, the spectral gap condition takes the form

$$\lambda_3 - \lambda_2 > 2. \tag{2}$$

The Routh–Hurwitz criterion for the instability of a point p gives

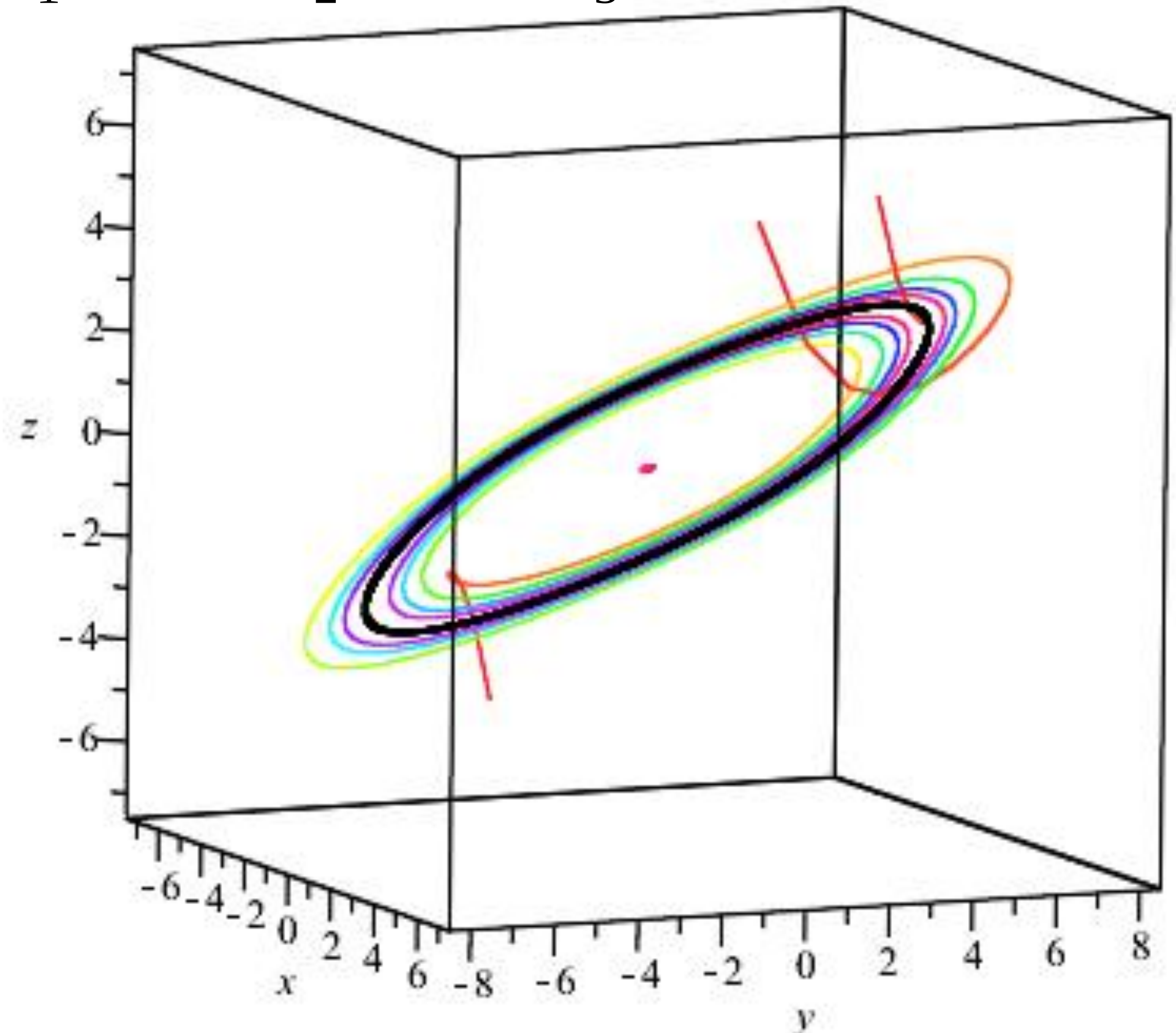
$$-g'(\nu) + \lambda_1\lambda_2\lambda_3 > (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3). \tag{3}$$

For a control function we can take, e.g.,

$$g(x_3) = \operatorname{arcctan}(x_3 - \nu), \quad \nu = \frac{\pi}{2\mu_1\mu_2\mu_3}; \tag{4}$$

then $g'(\nu) = -1$. Inequalities (2)–(3) determine a nonempty open set Ω in the positive octant $\mathbb{R}^3_+$ of $\lambda_1, \lambda_2, \lambda_3$. In particular, Ω contains all points of the form $(\varepsilon, \varepsilon, 2 + 2\varepsilon)$ for sufficiently small $\varepsilon > 0$. According to the main theorem, system (1) with $(\lambda_1, \lambda_2, \lambda_3) \in \Omega$ and control function (4) has a stable periodic mode.

Cycle: $\lambda_1 = 0.05, \lambda_2 = 0.05, \lambda_3 = 2.1$



A cellular dynamics model

Yet another example is related to a biochemical model [1]. This is the system

$$\begin{aligned}\dot{x} &= -kx + R(z)\\ \dot{y} &= x - G(y,z)\\ \dot{z} &= -qz + G(y,z)\end{aligned} \tag{1}$$

with $R(z) = \dfrac{1}{1+z^4}$, $G(y,z) = \dfrac{Ty(1+y)(1+z)^2}{L+(1+y)^2(1+z)^2}$, and constants $k, q > 0$, $T = 10$, and $L = 10^6$. Here $x, y, z > 0$ are dimensionless concentrations S_1, S_2, and S_3 of the initial, intermediate, and final substances, respectively; k and q are the change rate constants of S_1 and S_3.

We have $G_y > 0$, $G_z > 0$, and $\lim\limits_{y\to+\infty} G(y,z) = T$. Moreover,

$$-1.08 < R_z(z) \le 0, \quad G_y(y,z) < T, \quad G_z(y,z) < \frac{T}{2}.$$

LEMMA. *System* (1) *in* $\mathbb{R}^3_+$ *has at most one fixed point. If* $kT > 1$, *then the domain*

$$D:\ x_0 < x < \frac{1}{k},\ \ y_0 < y < y_1,\ \ 0 < z < \frac{T}{q},$$

where $x_0 = \dfrac{1}{k} R\left(\dfrac{T}{q}\right)$, $G\left(y_0, \dfrac{T}{q}\right) = x_0$, *and* $G(y_1, 0) = \dfrac{1}{k}$, *is positively invariant and contains the fixed point.*

Note. In the case $G\left(\bar{y}, \dfrac{T}{q}\right) = \dfrac{1}{k}$, since G increases in y and z, it follows that $y_0 < \bar{y} < y_1$.

1. C. Suguna, K.K. Chowdhury and S. Sinha, *Phys. Rev. E*, **60**:5 (1999).

Inertial manifold

In the natural decomposition $f=-A+F$ of the vector field f of system (1) into linear and nonlinear parts, we have

$$A=\begin{pmatrix}-k & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -q\end{pmatrix}, \qquad F\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}R(z)\\ x-G(y,z)\\ G(y,z)\end{pmatrix}.$$

The change $u=y+z$ reduces system (1) to the form

$$\begin{aligned}\dot{x}&=-kx+R(z)\\ \dot{u}&=x-qz\\ \dot{z}&=-qz+G(u-z,z)\end{aligned} \tag{2}$$

in the variables (x,u,z), and the vector field f_1 decomposes as $f_1=-A+F_1$, where

$$F_1\begin{pmatrix}x\\ u\\ z\end{pmatrix}=\begin{pmatrix}R(z)\\ x-qz\\ G(u-z,z)\end{pmatrix}.$$

Moreover,

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=C\begin{pmatrix}x\\ u\\ z\end{pmatrix}, \quad C=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}, \quad C^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

In what follows, to simplify formulations, we assume that $k>q$.

Inertial manifold. 1

We set $\lambda_1 = 0$, $\lambda_2 = q$, and $\lambda_3 = k$. The nonlinear component F_1 in system (2) is simpler than F. This allows us to reduce the upper bound $K = K(k,q)$ for the norm of its Jacobian matrix under the spectral gap condition $\lambda_3 - \lambda_2 > 2K$. Thereby, we extend the domain of parameters (k,q) ensuring the existence of a two-dimensional inertial manifold of system (2) and, hence, of system (1). If

$$\Delta:\ 0 < x < \frac{1}{k},\ 0 < y < y_1,\ 0 < z < \frac{T}{q}, \text{ where } G(y_1, 0) = \frac{1}{k},$$

then $\Delta \supset D$, and **the domain Δ narrows with increasing k and q.**
The constant $K = K(k,q)$, which is defined as

$$K = \max_{C^{-1}\Delta} \left\| F_1'(x,u,z) \right\|_2 = \max_{\Delta} \left\| (F_1' \circ C)(x,y,z) \right\|_2, \text{ where } F_1' \circ C = \begin{pmatrix} 0 & 0 & -R_z \\ 1 & 0 & -q \\ 0 & G_y & G_z - G_y \end{pmatrix}, \qquad (3)$$

can be estimated by using the inequality $\|B\|_2 \le \sqrt{\|B\|_\infty \cdot \|B\|_1}$, where $\|B\|_1 = \|B^*\|_\infty$, which is valid for arbitrary matrices B. Note that K **is always more than 1.** Employing reflections, we extend the function $F_1 \circ C$ defined on the rectangular cell Δ to a smooth function $F_2 : \mathbb{R}^3 \to \mathbb{R}^3$ without increasing K; then $F_2 \circ C^{-1}$ extends F_1 from the domain $C^{-1}\Delta$ to $\mathbb{R}^3$ with the same constant K.

Inertial manifold. 2

Let $\Theta=\{(k,q)\in\mathbb{R}_{+}^{2},\ k-q>2K(k,q)\}$; then system (1) has a two-dimensional inertial manifold, provided that $(k,q)\in\Theta$.

PROPOSITION. *If* $(k_0,q_0)\in\Theta$, *then* $(k,q)\in\Theta$ *for* $k\geq k_0,\ q\geq q_0$, *and* $k-q\geq k_0-q_0$.

Indeed, as k and q increase, the domain Δ narrows; therefore, the constant $K=K(k,q)$ in (3) does not increase, and the inequality $k-q>2K$ remains valid.

The existence of a two-dimensional inertial manifold provides certain information about the geometry of cycles. In the situation under consideration, any cycle Γ (if it exists) has the form of a graph

$$x=\sigma(u,z)=\sigma(y+z,z)=\sigma_1(y,z),$$

where $(x,y,z)\in\Gamma$ and σ is a smooth function on the plane of $(u,z)\in\mathbb{R}^2$.

Instability of fixed point

The Jacobian matrix of system (1) at a fixed point $p_s = (x_s, y_s, z_s)$ has the form

$$\begin{pmatrix} -k & 0 & -b \\ 1 & -c & -d \\ 0 & c & d-q \end{pmatrix},$$

where $b = -R_z(z_s)$, $c = G_y(y_s, z_s)$ and $d = G_z(y_s, z_s)$. The Routh–Hurwitz criterion says that the point p_s is unstable if and only if $a_1 < 0$ or $a_1 a_2 - a_3 < 0$, where

$$a_1 = c - d + k + q, \qquad a_2 = k(c-d) + qc + kq, \qquad a_3 = (kq+b)c\,.$$

As we see, the point p_s is unstable for $a_2 < 0$.

Final dynamics

The structure of the nonlinearity in the model under consideration is complicated; for this reason, we analyze the model by computational means. Using the Maple software, we show that system (1) admits asymptotically stable cycles. We estimate norms for points $p(x,y,z)\in\Lambda$.

For $k=3$ and $q=0.1$, we have $y_1\approx 186$,

$$\|F_1'(p)\|_\infty \le 1.209,\ \ \|F_1'(p)\|_1 \le 1.166,\ \ \|F_1'(p)\|_2 \le K = 1.187,$$

$$x_s \approx 0.117,\ \ y_s \approx 49.653,\ z_s \approx 1.167,$$

$$a_2 \approx -0.05.$$

For $k=2.5$ and $q=0.1$, we have $y_1\approx 204$,

$$\|F_1'(p)\|_\infty \le 1.209,\ \ \|F_1'(p)\|_1 \le 1.166,\ \ \|F_1'(p)\|_2 \le K = 1.187,$$

$$x_s \approx 0.123,\ \ y_s \approx 49.558,\ z_s \approx 1.230,$$

$$a_2 \approx -0.01.$$

In both cases, $a_2<0$ and $\lambda_3-\lambda_2>2K$; therefore, system (1) admits an asymptotically stable cycle.

Final dynamics. 1

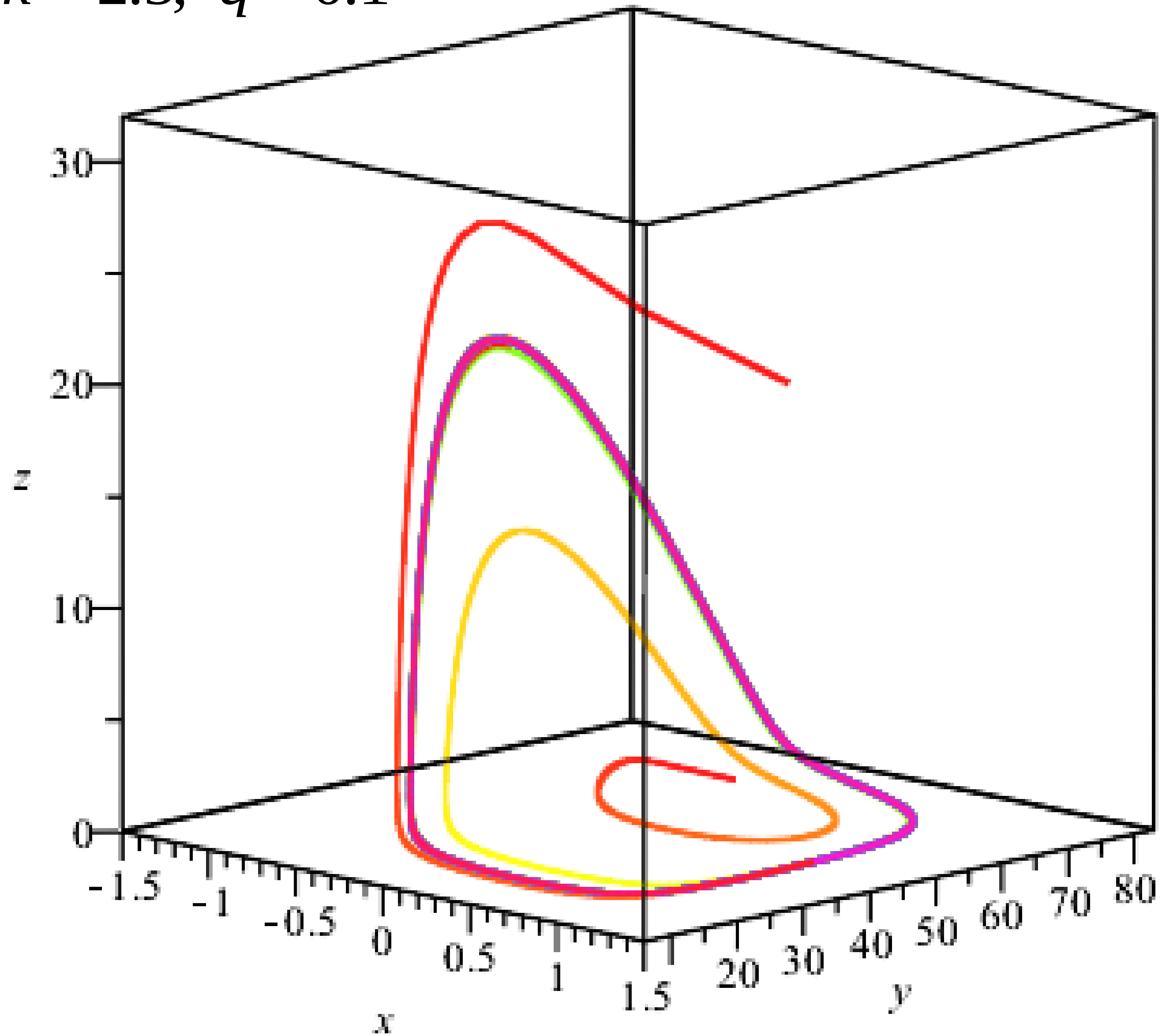
System (1) admits stable periodic modes in an open domain of parameters (k,q), which contains, for example, sufficiently small neighborhoods of the points $(3,0.1)$ and $(2.5,0.1)$.

System (1) demonstrates the two-dimensional final dynamics in a large domain Θ of parameters (k,q): if $(k_0,q_0)\in\Theta$ and

$$\Sigma(k_0,q_0)\doteq\{k\geq k_0,\ q\geq q_0,\ k-q\geq k_0-q_0\},$$

then the sector $\Sigma(k_0, q_0)$ is contained in Θ.

Cycle: $k = 2.5, \; q = 0.1$



Conclusion

Although its applicability domain is restricted, the spectral gap method well complements the list of known approaches to the problem of detecting asymptotically stable cycles of ODEs.

THANKS FOR ATTENTION