ISSN 0001-4346, Mathematical Notes, 2018, Vol. 103, No. 5, pp. 811–819. © Pleiades Publishing, Ltd., 2018.

The Jordan Property for Lie Groups and Automorphism Groups of Complex Spaces*

V. L. Popov[1**]

[1] *Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

Received April 3, 2018

Abstract—We prove that the family of all connected n-dimensional real Lie groups is uniformly Jordan for every n. This implies that all algebraic (not necessarily affine) groups over fields of characteristic zero and some transformation groups of complex spaces and Riemannian manifolds are Jordan.

DOI: 10.1134/S0001434618050139

Keywords: *Jordan group, bounded group, Lie group, algebraic group, automorphism group of complex space, isometry group of Riemannian manifold.*

1. INTRODUCTION

We recall the definition introduced in [1, Def. 2.1].

Definition 1. Given a group G, let

$$J_G := \sup_F \min_A [F : A],$$

where F runs over all finite subgroups of G and A runs over all normal Abelian subgroups of F. If $J_G \neq \infty$, then G is called a *Jordan group*, and J_G is called the *Jordan constant* of G. In this case, we also say that G has the *Jordan property*.

Informally, the Jordan property of G means that all finite subgroups of G are "almost Abelian" in the sense that they are extensions of Abelian groups by groups from a finite list. Definition 1 is inspired by the classical theorem of Jordan [2] claiming that $J_{\mathrm{GL}_n(\ell)} \neq \infty$ holds for every n and every field ℓ of characteristic zero. If ℓ is algebraically closed, then, for every fixed n, the constant $J_{\mathrm{GL}_n(\ell)}$ is independent of ℓ, and we denote it simply by $J(n)$. It has been computed in [3]; in particular,

$$J(n) = (n+1)! \qquad \text{for all } n \geq 71 \text{ and } n = 63, 65, 67, 69.$$

For more examples of Jordan groups, see [4].

In what follows, by a variety we mean an algebraic variety over a fixed algebraically closed field k of characteristic zero; in particular, any algebraic group is defined over k. If G is either an algebraic group or a topological group, G^0 denotes the identity component of G.

The following problem was posed seven years ago in [1, Sec. 2] (see also [4, Sec. 2]); since then, it has been explored by a number of researchers (see the most recent brief survey and references in [5, Sec. 1]).

Problem. *Describe varieties X for which the group* $\mathrm{Aut}(X)$ *is Jordan.*

At present (April 2018), it is still unknown whether there are varieties X such that the group $\mathrm{Aut}(X)$ is non-Jordan (note that complex manifolds whose automorphism groups are non-Jordan do exist; see Remark 2 below). On the other hand, for many types of varieties X, it has been shown that the group $\mathrm{Aut}(X)$ is Jordan. In particular, Sheng Meng and De-Qi Zhang have recently proved the following theorem.

*The article was submitted by the author for the English version of the journal.

**E-mail: popovvl@mi.ras.ru

Theorem 1 ([6, Therem 1.6]). *For every projective variety X, the group $\mathrm{Aut}(X)$ is Jordan.*

Given a variety X, by $\mathrm{Aut}(X)^0$ we denote the identity component of $\mathrm{Aut}(X)$ in the sense of [7]; see also [8]. According to [7, Cor. 1], if X is complete, then $\mathrm{Aut}(X)^0$ is a connected (not necessarily affine) algebraic group. Jordan's theorem cited above implies that every affine algebraic group is Jordan; see [4, Theorem 2]. The key ingredient of the proof of Theorem 1 given in [6] is the proof that the extension of this claim to all (i.e., not necessarily affine) algebraic groups holds true. The latter proof is rather involved.

In the present note, we obtain a general result (with a very short proof), from which the above-mentioned extension immediately follows (see Theorem 4 below). Namely, we prove that every finite-dimensional connected real Lie group is Jordan (more precise and general statements are formulated in Theorems 2 and 3 and Corollary 3 below). Then, in Secs. 5–7, we apply this result to show that certain transformation groups of complex spaces and Riemannian manifolds are Jordan (see Theorems 5, 7–9, and 10 below).

2. LIE GROUPS

We now explore the Jordan property for finite-dimensional real Lie groups G. Note that non-Jordan groups of this type do exist, because every discrete group is a 0-dimensional real Lie group and there are non-Jordan discrete groups (see [4, 1.2.5]). Therefore, G can be expected to be Jordan only under some restriction on the component group G/G^0.

To formulate this restriction, we recall the following definition introduced in [1, Def. 2.9].

Definition 2. Given a group H, we set

$$b_H := \sup_F |F|,$$

where F runs over all finite subgroups of H. If $b_H \neq \infty$, then the group H is said to be *bounded*.

In particular, every finite group H is bounded, and $b_H = |H|$.

In Theorems 2 and 3 and Corollary 1 below, we consider the class of finite-dimensional real Lie groups G whose component group G/G^0 is bounded. Note that every compact Lie group K belongs to this class, because K/K^0 is finite.

Theorem 2. *Let G be a finite-dimensional real Lie group whose component group G/G^0 is bounded. Then G is Jordan.*

Proof. In view of [1, Lemma 2.11] (or [4, Therem 5]), we may (and shall) assume that G is connected. This assumption implies the existence of a compact Lie subgroup K of G such that every compact subgroup of G is conjugate to that of K (see, e.g., [10, Chap. XV, Theorem 3.1(iii)]). In particular, every finite subgroup of G is conjugate to that of K. This and Definition 1 show that G is Jordan if and only if so is K, and if they are, then

$$J_G = J_K. \tag{1}$$

Being compact, the group K admits a faithful finite-dimensional representation, i.e., is isomorphic to a subgroup of $\mathrm{GL}_m(\mathbb{R})$ for some m (see, e.g., [11, Chap. 5, Sec. 2, Theorem 10]). Since the latter group is Jordan, K is Jordan as well (see [4, Theorem 3(i)]. This completes the proof. □

Corollary 1. *For every finite-dimensional real Lie group G whose component group G/G^0 is bounded, the set of isomorphism classes of all finite simple subgroups of G is finite.*

We now dwell on estimating the Jordan constants of Lie groups whose component group is finite, with a view to proving that the class of such groups has a property stronger than that all of its members are Jordan (see Corollary 3 below). Seeking only this goal, we did not seek to improve the estimates obtained.

Lemma 1. *Let S be a simply connected simple affine algebraic group. Then the minimum* $\operatorname{rdim} S$ *of dimensions of faithful linear algebraic representations of S is given by the following table*:

type of S	A_ℓ $_{\ell\geq 1}$	B_ℓ $_{\ell\geq 2}$	C_ℓ $_{\ell\geq 2}$	D_ℓ $_{\ell\geq 3,\ \ell \text{ odd}}$	D_ℓ $_{\ell\geq 4,\ \ell \text{ even}}$	E_6	E_7	E_8	F_4	G_2
$\operatorname{rdim} S$	$\ell+1$	2^ℓ	2ℓ	$2^{\ell-1}$	$2\ell+2^{\ell-1}$	27	56	248	26	7

Remark 1. In the proof of Lemma 1 below, a faithful representation of S of dimension $\operatorname{rdim} S$ is explicitly specified for each type of S.

Proof of Lemma 1. By Lefschetz's principle (see, e.g., [12, VI.6]), we may (and shall) assume that k is $\mathbb{C}$. We fix a maximal torus T of S. Let $\alpha_1, \dots, \alpha_\ell \in (\operatorname{Lie} T)^*$, $\varpi_1, \dots, \varpi_\ell \in (\operatorname{Lie} T)^*$, and $\alpha_1^\vee, \dots, \alpha_\ell^\vee \in \operatorname{Lie} T$ be, respectively, the systems of simple roots, fundamental weights, and simple coroots of $\operatorname{Lie} T$ with respect to a fixed Borel subalgebra of $\operatorname{Lie} S$ containing $\operatorname{Lie} T$; we number them as in [11].

The center Z of S is a finite subgroup of T. Fix a subset $\widetilde{Z}$ of $\operatorname{Lie} T$ whose image under the exponential map $\operatorname{Lie} T \to T$ is the set of all nonidentity elements of Z.

For every dominant weight $\lambda \in (\operatorname{Lie} T)^*$, let $R(\lambda)$ be an irreducible representation of $\operatorname{Lie} S$ with the highest weight λ. The dimension of $R(\varpi_i)$ for every i is specified in [11, Ref. Chap., Sec. 2, Table 5, pp. 299–305]. Note that Weyl's dimension formula implies

$$\dim R(\textstyle\sum_{i=1}^{\ell} \lambda_i \varpi_i) \geq \dim R(\sum_{i=1}^{\ell} \mu_i \varpi_i) \qquad \text{if } \lambda_i \geq \mu_i \text{ for every } i. \tag{2}$$

Since S is simply connected, $R(\lambda)$ is the differential of a finite-dimensional linear algebraic representation $\mathcal{R}(\lambda)$ of S. Since S is simple, for every finite set D of nonzero dominant weights and $\mathcal{R}(D) := oplus_{\lambda\in D}\mathcal{R}(\lambda)$, we have $\ker \mathcal{R}(D) \subseteq Z$. Hence

$$\mathcal{R}(D) \text{ is faithful} \iff \text{for every } x \in \widetilde{Z}, \text{ there is a } \lambda \in D \text{ with } \lambda(x) \notin \mathbb{Z}. \tag{3}$$

As is well known, $\dim \mathcal{R}(\varpi_1)$ is the minimal dimension of nonzero finite-dimensional algebraic representations of S (see [11, pp. 299–305]).

If S is of type E_8, F_4, or G_2, then Z is trivial; hence, in this case, $\mathcal{R}(\varpi_1)$ is faithful and, therefore, we have the equality

$$\operatorname{rdim} S = \dim R(\varpi_1), \tag{4}$$

which proves Lemma 1 for these types.

If S is of type A_ℓ or C_ℓ, then S is $\mathrm{SL}_{\ell+1}$ or $\mathrm{Sp}_{2\ell}$, respectively. For these groups, $\mathcal{R}(\varpi_1)$ is the tautological faithful representation; therefore, in this case, (4) holds as well, which proves Lemma 1 for these types.

For the other types, we apply (3) to the set $\widetilde{Z}$ taken from [11, Ref. Chap., Sec. 2, Table 3, p. 298]. In what follows, we use the fact that, for any $\lambda_i, \mu_i \in k$,

$$\text{the value of } \textstyle\sum_{i=1}^{\ell} \lambda_i \varpi_i \in (\operatorname{Lie} T)^* \text{ in } \sum_{i=1}^{\ell} \mu_i \alpha_i^\vee \in \operatorname{Lie} T \text{ is } \sum_{i=1}^{\ell} \lambda_i \mu_i. \tag{5}$$

If S is of type E_7, then $\widetilde{Z}$ consists of only one element $\zeta := (\alpha_1^\vee + \alpha_3^\vee + \alpha_7^\vee)/2$. By (5), we have $\varpi_1(\zeta) = 1/2 \notin \mathbb{Z}$, so that $\mathcal{R}(\varpi_1)$ is faithful. Therefore, in this case, (4) holds too, which proves Lemma 1 for this type.

If S is of type E_6, then $\widetilde{Z}$ consists of the two elements $\zeta := (\alpha_1^\vee - \alpha_2^\vee + \alpha_4^\vee - \alpha_5^\vee)/3$ and 2ζ. Since $\varpi_1(z) = 1/3 \notin \mathbb{Z}$ and $\varpi_1(2z) = 2/3 \notin \mathbb{Z}$, in this case, $\mathcal{R}(\varpi_1)$ is again faithful, which implies (4). This proves Lemma 1 for this type.

If S is of type B_ℓ, then $\widetilde{Z}$ consists of only one element $\alpha_\ell^\vee/2$. This and (3), (5) imply that $\mathcal{R}(D)$ is faithful if and only if D contains $\sum_{i=1}^{\ell} \lambda_i \varpi_i$ with odd λ_ℓ. Using (2), from this we infer that $\mathcal{R}(\varpi_\ell)$ is the faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_\ell)$. This proves Lemma 1 for this type.

If S is of type D_ℓ, $\ell \geq 3$, ℓ odd, then $\widetilde{Z}$ consists of three elements

$$\zeta := (\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{\ell-2}^\vee)/2 + (\alpha_{\ell-1}^\vee - \alpha_\ell^\vee)/4, \quad 2\zeta, \quad 3\zeta. \tag{6}$$

From (3), (5), and (6) we infer that $\mathcal{R}(D)$ is faithful if and only if D contains $\sum_{i=1}^{\ell} \lambda_i \varpi_i$ such that 4 is coprime to either $\lambda_{\ell-1}$ or λ_ℓ. This and (2) show that $\mathcal{R}(\varpi_\ell)$ is a faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_\ell)$, which proves Lemma 1 for this type.

If S is of type D_ℓ for even $\ell \geq 4$, then $\widetilde{Z}$ consists of the three elements

$$\zeta_1 := (\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{\ell-1}^\vee)/2, \quad \zeta_2 := (\alpha_{\ell-1}^\vee + \alpha_\ell^\vee)/2, \quad \zeta_1 + \zeta_2. \tag{7}$$

Hence if $\mathcal{R}(D)$ is faithful, then D contains $\sum_{i=1}^{\ell} \lambda_i \varpi_i$ with odd λ_ℓ or $\lambda_{\ell-1}$ and $\sum_{i=1}^{\ell} \mu_i \varpi_i$ with odd μ_i for some odd $i \neq \ell - 1$. On the other hand, since Z is not cyclic in this case, Schur's lemma implies $|D| \geq 2$. From this it is not difficult to deduce that $\mathcal{R}(\varpi_1) \oplus \mathcal{R}(\varpi_\ell)$ is a faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_1) + \dim R(\varpi_\ell) = 2\ell + 2^{\ell-1}$. This completes the proof of Lemma 1. □

Corollary 2. *Every simply connected simple affine algebraic group of rank ℓ admits a faithful linear algebraic representation of dimension at most $2^\ell + 10$.*

Proof. Clearly, if an algebraic group admits a faithful linear algebraic representation, then it admits a faithful linear algebraic representation of any bigger dimension. In view of this, the claim follows from the inequality $\operatorname{rdim} S \leq 2^\ell + 10$, which, in turn, follows from Lemma 1: indeed, the latter shows that $\operatorname{rdim} S \leq 2^\ell$ if the type of S differs from F_4 and G_2, and that $\operatorname{rdim} S = 2^\ell + 10$ and $2^\ell + 3$, respectively, for the types F_4 and G_2. □

Theorem 3. *Let G be an n-dimensional real Lie group whose component group G/G^0 is bounded. Then*

$$J_G \leq b_{G/G^0} J(n(2^n + 10))^{b_{G/G^0}}. \tag{8}$$

Proof. According to [1, Lemma 2.11] (or [4, Theorem 5]), we may (and shall) assume that G is connected; in particular,

$$b_{G/G^0} = 1. \tag{9}$$

We use the notation of the proof of Theorem 2. Since G is connected, it follows that K is connected, too; see [10, Chap. XV, Theorem 3.1(ii)]. Hence (see [13, Sec. 1, Proposition 4]) there are

(i) compact simply connected simple Lie groups $K_1, \dots, K_d$;

(ii) a compact torus S;

(iii) a group epimorphism with finite kernel

$$\pi\colon \widetilde{K} := K_1 \times \cdots \times K_d \times S \to K. \tag{10}$$

Using Theorem 3 (ii) of [4], from (iii) we infer

$$J_K \leq J_{\widetilde{K}}. \tag{11}$$

Every K_i is a real form of the corresponding simply connected simple complex affine algebraic group. The rank ℓ_i of the latter is equal to that of K_i. In view of Corollary 2, we conclude that K_i admits an embedding in $\mathrm{GL}_{2^{\ell_i}+10}(\mathbb{C})$. Since $\ell_i \leq \dim \widetilde{K} = \dim K \leq n$, this, in turn, implies that K_i admits an embedding in $\mathrm{GL}_{2^n+10}(\mathbb{C})$. Clearly, S admits an embedding in $\mathrm{GL}_{\dim S}(\mathbb{C})$, and therefore, since $\dim S \leq \dim \widetilde{K}$, in $\mathrm{GL}_{2^n+10}(\mathbb{C})$. This and the definition of $\widetilde{K}$ (see (10)) show that $\widetilde{K}$ admits an embedding in the direct product of $d+1$ copies of $\mathrm{GL}_{2^n+10}(\mathbb{C})$ and hence in $\mathrm{GL}_{(d+1)(2^n+10)}(\mathbb{C})$. Since $d+1 \leq \dim \widetilde{K}$ in view of (10), it follows that $\widetilde{K}$ admits an embedding in $\mathrm{GL}_{n(2^n+10)}(\mathbb{C})$; hence

$$J_{\widetilde{K}} \leq J(n(2^n+10)). \tag{12}$$

Now, collecting (1), (11), (12), and (9) together, we complete the proof. □

We recall the following definition from [6].

Definition 3. A family $\mathcal{F}$ of groups is said to be *uniformly Jordan* if every group in $\mathcal{F}$ is Jordan and there is an integer $J_{\mathcal{F}}$ such that $J_G \leq J_{\mathcal{F}}$ for every $G \in \mathcal{F}$.

Corollary 3. *Fix an integer $n \geq 0$. Let $\mathcal{L}_n$ be the family of all connected n-dimensional real Lie groups. Then*

(i) *the family $\mathcal{L}_n$ is uniformly Jordan*;

(ii) *one can take $J_{\mathcal{L}_n} = J(n(2^n+10))$.*

Proof. This assertion follows from (8), because $b_{G/G^0} = 1$ for every $G \in \mathcal{L}_n$. □

Corollary 4. *For every integer $n \geq 0$, the set of isomorphism classes of finite simple groups embeddable in n-dimensional connected real Lie groups is finite.*

3. ALGEBRAIC GROUPS

Now we consider several applications of Theorems 2 and 3. First, we apply them to algebraic groups, answering Question 1.2 in [6].

Theorem 4. *Every (not necessarily affine) n-dimensional algebraic group G over an algebraically closed field k of characteristic 0 is Jordan. Moreover,*

$$J_G \leq [G : G^0] J(n(2^{2n+1}+20))^{[G:G^0]}. \tag{13}$$

Proof. In the case under consideration, G/G^0 is finite. By Lefschetz's principle, we may (and shall) assume that k is $\mathbb{C}$. Then G has the structure of a $2n$-dimensional real Lie group whose identity component is G^0. The required assertion then follows from Theorem 3. □

Statement (i) of the next corollary is one of the main results of [6].

Corollary 5. *Fix an integer $n \geq 0$. Let $\mathcal{A}_n$ be the family of all (not necessarily affine) connected n-dimensional algebraic groups over an algebraically closed field k of characteristic 0. Then*

(i) ([6, Theorem 1.3]) *the family $\mathcal{A}_n$ is uniformly Jordan*;

(ii) *one can take $J_{\mathcal{A}_n} = J(n(2^{2n+1}+20))$.*

Proof. This follows from (13). □

4. AUTOMORPHISM GROUPS OF COMPLEX SPACES

The next application is to automorphism groups of complex spaces.

Let C be a (not necessarily reduced) complex space. There exists a topology on $\mathrm{Aut}(C)$ with respect to which $\mathrm{Aut}(C)$ is a topological group (see [14, 2.1]).

Theorem 5. *For every compact complex space C, the group $\mathrm{Aut}(C)^0$ is Jordan.*

Proof. According to [15], the compactness of C implies that $\mathrm{Aut}(C)$ is a complex Lie group. The required assertion then follows from Theorem 2. □

We do not know whether the statement of Theorem 5 remains true if $\mathrm{Aut}(C)^0$ is replaced by $\mathrm{Aut}(C)$. According to [5, Theorem 1.5], the answer is affirmative if C is a connected compact two-dimensional complex manifold. By Theorem 1, it is also affirmative if C is a projective variety. On the other hand, we recall that, by [16], there are connected smooth compact real manifolds whose diffeomorphism groups are non-Jordan (this disproves Ghys' conjecture).

Remark 2. There are connected noncompact complex manifolds whose automorphism groups are non-Jordan. Indeed, according to [17], for any countable group Γ, there is a noncompact Riemann surface M such that $\mathrm{Aut}(M)$ is isomorphic to Γ; this implies the claim because of the existence of countable non-Jordan groups (see [4, Sec. 1.2.5]).

In fact, using the idea exploited earlier in [18], one can prove more than said in Remark 2, showing the existence of connected complex manifolds with monstrous automorphism groups; namely, the following theorem is valid.

Theorem 6. *There is a 3-dimensional simply connected noncompact complex manifold M such that*

(i) *the group $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presentable (in particular, every finite) group*;

(ii) *every such copy is a discrete transformation group of M acting freely.*

Proof. It follows (see, e.g., [19, Theorem 12.29]) from Higman's embedding theorem [20] that there is a universal finitely presented group, i.e., a finitely presented group $\mathcal{U}$ containing, as a subgroup, an isomorphic copy of every finitely presented group. In turn, according to [21, Corollary 1.66], the finite presentability of $\mathcal{U}$ implies the existence of a connected 3-dimensional compact complex manifold B whose fundamental group is isomorphic to $\mathcal{U}$. Let $\pi\colon \widetilde{B} \to B$ be the universal cover. Then $\widetilde{B}$ is a simply connected noncompact 3-dimensional complex manifold, and the deck transformation group of π is a subgroup of $\mathrm{Aut}\,\widetilde{B}$ isomorphic to $\mathcal{U}$, which acts on $\widetilde{B}$ freely. Hence one can take $M = \widetilde{B}$. □

Remark 3. For M from Theorem 6, the group $\mathrm{Aut}(M)$ is non-Jordan, because for every integer n, there is a finite simple group of order $> n$ (cf. [4, Example 4].

Theorem 7. *Fix an integer $n \geq 0$. Let $\mathcal{C}_n$ be the family of groups $\mathrm{Aut}(M)^0$, where M runs over all connected compact complex manifolds of complex dimension n. Then*

(i) *the family $\mathcal{C}_n$ is uniformly Jordan*;

(ii) *one can take $J_{\mathcal{C}_n} = J((2n^2+n)(2^{2n^2+n}+10))$.*

Proof. For $G := \mathrm{Aut}(M)^0$, let K be as in the proof of Theorem 2. By Montgomery–Zippin's theorem, $\dim K \leq 2n^2 + n$. Since, clearly, $J(m)$ is a nondecreasing function of m, the latter inequality, (1), and Theorem 3 yield $J_G \leq J((2n^2+n)(2^{2n^2+n}+10))$. This proves (i) and (ii). □

5. AUTOMORPHISM GROUPS OF HYPERBOLIC COMPLEX MANIFOLDS

The next application is to complex manifolds hyperbolic in the sense of Kobayashi (in particular, to bounded domains in $\mathbb{C}^n$).

Theorem 8. *Fix an integer $n \geq 0$. Let $\mathcal{H}_n$ be the family of groups $\mathrm{Aut}(M)^0$, where M runs over all connected complex manifolds hyperbolic in the sense of Kobayashi and of complex dimension n. Then*

(i) *the family $\mathcal{H}_n$ is uniformly Jordan*;

(ii) *one can take $J_{\mathcal{H}_n} = J((2n+n^2)(2^{2n+n^2}+10))$*;

(iii) *for every point $x \in M$, the $\mathrm{Aut}(M)$-stabilizer $\mathrm{Aut}(M)_x$ of x is Jordan and $J_{\mathrm{Aut}(M)_x} \leq J(n)$.*

Proof. Let M be a connected complex manifold hyperbolic in the sense of Kobayashi and of complex dimension n. By [22, Theorems 2.1 and 2.6], $\mathrm{Aut}(M)$ is a real Lie group of dimension $\leq 2n+n^2$; hence (i) and (ii) follow by Theorems 2 and 3. According to [22, Theorem 2.6], the isotropy representation of $\mathrm{Aut}(M)_x$ is faithful, and its image is isomorphic to a subgroup of the unitary group $\mathrm{U}(n)$, which implies (iii). □

Remark 4. The group $\mathrm{Aut}(M)^0$ in the formulation of Theorem 8 cannot be replaced by $\mathrm{Aut}(M)$. Indeed, it follows from the construction in [17] that the Riemann surface M in Remark 2 is hyperbolic in the sense of Kobayashi. Hence there are connected hyperbolic complex manifolds M such that the group $\mathrm{Aut}(M)$ is not Jordan.

However, as the next theorem shows, for complex hyperbolic manifolds M of a special type, the Jordan property holds for the whole $\mathrm{Aut}(M)$ rather than only for $\mathrm{Aut}(M)^0$.

Theorem 9. *For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the group $\mathrm{Aut}(M)$ of all biholomorphic transformations of M is Jordan.*

Proof. If the Lie group $\mathrm{Aut}(M)$ is compact, then the claim follows from Theorem 2. If the group $\mathrm{Aut}(M)$ is noncompact, then, by the Rosey–Wong theorem [23], [24], the domain M is biholomorphic to the unit ball B_n in $\mathbb{C}^n$. Since $\mathrm{Aut}(B_n)$ is $\mathrm{PU}(n,1)$ (see [14, Sec. 2.7, Proposition 3]) and the latter Lie group is connected (see [25, Chap. IX, Lemma 4.4]), so that the required assertion then follows from Theorem 2. □

Corollary 6. *For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the set of isomorphism classes of all finite simple groups of biholomorphic transformations of M is finite.*

6. ISOMETRY GROUPS OF RIEMANNIAN MANIFOLDS

The last application is to isometry groups $\mathrm{Iso}(M)$ of Riemannian manifolds M. They are topological groups with respect to the compact-open topology [26].

Theorem 10. *Fix an integer $n \geq 0$. Let $\mathcal{R}_n$ be the family of groups $\mathrm{Iso}(M)^0$, where M runs over all connected n-dimensional Riemannian manifolds. Then*

(i) *the family $\mathcal{R}_n$ is uniformly Jordan;*

(ii) *one can take $J_{\mathcal{R}_n} = J((n^2+n)(2^{(n^2+n-2)/2}+5))$*;

(iii) *for every point $x \in M$, the $\mathrm{Iso}(M)$-stabilizer $\mathrm{Iso}(M)_x$ of x is Jordan*;

(iv) *if the manifold M is compact, then the group $\mathrm{Iso}(M)$ is Jordan.*

Proof. It is known (see, e.g., [26, Chap. II, Theorems 1.2 and 3.1]) that $\mathrm{Iso}(M)$ is a real Lie group of dimension at most $n(n+1)/2$, the group $\mathrm{Iso}(M)_x$ is compact for every x, and the group $\mathrm{Iso}(M)$ is compact if the manifold M is compact. The claim then follows by combining these facts with Theorems 2 and 3. $\square$

Remark 5. The group $\mathrm{Aut}(M)^0$ in the statement of Theorem 10 cannot be replaced by $\mathrm{Aut}(M)$. Indeed, it follows from the construction in [17] that the Riemann surface M in Remark 2 is a two-dimensional Riemannian manifold and $\mathrm{Aut}(M) = \mathrm{Iso}(M)$. Hence there are connected Riemannian manifolds M such that the group $\mathrm{Iso}(M)$ is not Jordan.

CONCLUDING REMARKS

1. In view of (1), computing the Jordan constants of connected real Lie groups reduces to that of such compact groups. For instance, results of [3] can be interpreted as the computation of the Jordan constants of all unitary groups:

$$J_{\mathrm{U}_n} = J(n) \qquad \text{for every } n.$$

Discussion in Sec. 2 leads to the following natural problem.

Problem. *Compute the Jordan constants of all simple compact connected real Lie groups.*

2. We expect that the topic of this note may be related to that of [18], [27], [28], [5], [29], [30], and [31].

ACNOWLEDGEMENTS

I was asked the question of whether the Lie groups are Jordan by A. M. Vershik (see [9, 95:20]), whom I thank. I am grateful to Yu. G. Zarhin for valuable comments.

This work was carried out at the Steklov Mathematical Institute and supported by the Russian Science Foundation under grant 14-50-00005.

REFERENCES

1. V. L. Popov, "On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties," in *Affine Algebraic Geometry*: *The Russell Festschrift* (CRM Proceedings and Lecture Notes, American Math. Society, Providence, RI, 2011), Vol. 54, pp. 289–311.
2. C. Jordan, "Mémoire sur les equations différentielle linéaire à intégrale algébrique," J. Reine Angew. Math. **84**, 89–215 (1878).
3. M. J. Collins, "On Jordan's theorem for complex linear groups," J. Group Theory **10**, 411–423 (2007).
4. V. L. Popov, "Jordan groups and automorphism groups of algebraic varieties," in *Automorphisms in Birational and Affine Geometry* (*Levico Terme, Italy, October* 2012|*rm)*, in *Springer Proceedings in Mathematics & Statistics,* (Springer, Heidelberg, 2014), Vol. 79, pp. 185–213.
5. Yu. Prokhorov and C. Shramov, "Automorphism groups of compact complex surfaces," `arXiv:1708. 03566` (2017).
6. S. Meng and D.-Q. Zhang, "Jordan property for non-linear algebraic groups and projective varieties," to appear in Amer. J. Math., available at `https://Preprint.press.jhu.edu/ajm/article/jordan-property-non-linear-algebraic-groups-and-projective-varieties`.
7. C. P. Ramanujam, "A note on automorphism groups of algebraic varieties," Math. Ann. **156**, 25–33 (1964).
8. V. L. Popov, "On infinite dimensional algebraic transformation groups," Transform. Groups **19** (2), 549–568 (2014).
9. V. L. Popov, "Jordan groups", talk at the St. Petersburg Division of Steklov Institute of Mathematics, Russian Academy of Sciences, December 18, 2014, videorecord available at `http://www.mathnet.ru/php/seminars.phtml?&presentid=10616&option_lang=eng`.
10. G. Hochschild, *The Structure of Lie Groups* (Holden-Day, San Francisco, 1965).
11. A. L. Onishchik, È. B. Vinberg, *Lie Groups and Algebraic Groups* in *Springer Series in Soviet Mathematics* (Springer-Verlag, Berlin, 1990).
12. J. H. Silverman, *The Arithmetic of Elliptic Curves* in *Graduate Texts in Math.* (2nd ed., Springer, Dordrecht, 2009), Vol. 106.
13. N. Bourbaki, *Groupes et Algèbres de Lie. Groupes de Lie Réels Compacts* (Masson, Paris, 1982), Chap. 9.

14. D. N. Akhiezer, *Lie Group Actions in Complex Analysis* in *Aspects of Mathematics* (Vieweg, Braunschweig, 1995), Vol. E 27.
15. W. Kaup, "Infinitesimale Transformationsgruppen komplexer Räume," Math. Ann. **160**, 72–92 (1965).
16. B. Csikós, L. Pyber, and E. Szabó, "Diffeomorphism groups of compact 4-manifolds are not always Jordan," `arXiv:1411.7524` (2014).
17. J. Winkelmann, "Realizing countable groups as automorphism groups of Riemann surfaces," Documenta Math. **7**, 413–417 (2002).
18. V. L. Popov, "Finite subgroups of diffeomorphism groups," Trudy MIAN **289**, 235–241 (2015) [Proc. Steklov Inst. Math. **289**, 221–226 (2015)].
19. J. J. Rotman, *An Introduction to the Theory of Groups* in *Graduate Texts in Math.* (4th ed., Springer, New York, 1995), Vol. 148.
20. G. Higman, "Subgroups of finitely presented groups," Proc. R. Soc. London A **262**, 455–475 (1961).
21. J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo, *Fundamental Groups of Compact Kähler Manifolds* in *Math. Surveys and Monographs* (American Mathematical Society, Providence, RI, 1996), Vol. 44.
22. S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings. An Introduction* (2nd ed., World Scientific, New Jersey, 2005).
23. J.-P. Rosay, "Sur une caractérisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes," Ann. Inst. Fourier (Grenoble) **29** (4), 91–97 (1979).
24. B. Wong, "Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group," Invent. math. **41**, 253–257 (1977).
25. S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press, New York, 1962).
26. S. Kobayashi, *Transformation Groups in Differential Geometry* (Springer, Berlin, 1995).
27. V. V. Przyjalkowski and C. A. Shramov, "Double quadrics with large automorphism groups," Trudy MIAN **294**, 167–190 (2016) [Proc. Steklov Inst. Math. **294**, 154–175 (2016)].
28. Yu. Prokhorov and C. Shramov, "Jordan constant for Cremona group of rank 3," Mosc. Math. J. **17** (3), 457–509 (2017).
29. Yu. G. Prokhorov, "On G-Fano threefolds," Izv. Ross. Akad. Nauk Ser. Mat. **79** (4), 159–174 (2015). [Russian Acad. Sci. Izv. Math. **79** (4), 795–808 (2015)].
30. Vik. S. Kulikov and E. I. Shustin, "On G-rigid surfaces," Trudy MIAN **298**, 144–164 (2017) [Proc. Steklov Inst. Math. **298**, 133–151 (2017)].
31. V. L. Popov, "Borel subgroups of Cremona groups," Mat. zametki **102** (1), 72–80 (2017) [Math. Notes **102** (1), 60–67 (2017)].